QUIZ 3

NAME:

Problem 1. A cannonball is shot out of a cannon, and due to unusual weather conditions, flys along the abnormal path ($\sqrt{(1-x^2)}$) How long is the cannonball's path through the sky? (Assume that the ground is given by $y = 0$.)

Solution: Note: this problem was too difficult. For grading, I only looked to see if you had reached the step marked ($\star$).
We use the formula

$$\ell = \int_a^b ds$$

where $ds = \sqrt{1 + (f'(x))^2}dx$. The derivative here is $(100 - 2x)$, making our integral($\star$).

$$\ell = \int_0^{100} \sqrt{1 + (100 - 2x)^2}dx$$

If you got to this part, you got full credit. To actually solve the problem was a little more difficult. Making the substitution $r = 100 - 2x$, we have $dr = -2dx$

$$\ell = \int_a^b \frac{-1}{2}\sqrt{1 + (r)^2}du$$

Making the substitution $r = \tan\theta$, $dr = \sec^2\theta d\theta$

$$\frac{-1}{2}\int_a^b \sec^3\theta d\theta$$

From here, there were two ways to solve the problem. One uses a substitution of sinh. However, as this was only briefly covered in the textbook, we will solve by integrating by parts.

$$\begin{aligned}
\int \sec^3\theta\, d\theta &= \int u\, dv \\
&= uv - \int v\, du \\
&= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta \\
&= \sec\theta\tan\theta - \int \sec\theta\,(\sec^2\theta - 1)\, d\theta \\
&= \sec\theta\tan\theta - \left(\int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta.\right) \\
&= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta.
\end{aligned}$$

Next we add $\int \sec^3 x\, dx$ to both sides of the equality just derived:

$$\begin{aligned}
2\int \sec^3\theta\, d\theta &= \sec\theta\tan\theta + \int \sec\theta\, d\theta \\
&= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C.
\end{aligned}$$

Then divide both sides by 2:

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C_1.$$

Problem 2. [Discussion 214] Determine whether or not the integral is convergent or divergent. If it is convergent, evaluate it. If it is divergent, prove that it diverges.

$$\int_0^\infty (\sin x)^2 e^{-x} dx$$

Solution: First, let's show that this integral converges. We know that $(\sin x)^2 e^{-x} \leq e^{-x}$, and $\int_0^\infty e^{-x} dx = 1$, so this must converge by the comparison test. Now let us actually compute what it converges to.

$$\int_0^\infty (\sin x)^2 e^{-x} dx = \lim_{m \to \infty} \int_0^m (\sin x)^2 e^{-x} dx$$

Integrating by parts, we let $u = (\sin x)^2$ and $dv = e^{-x}$

$$= \lim_{m \to \infty} \left(-(\sin x)^2 e^{-x} \Big|_0^m - \int_0^m 2\sin(x)\cos(x) e^{-x} dx \right)$$
$$= \lim_{m \to \infty} \left(-(\sin x)^2 e^{-x} \Big|_0^m - \int_0^m \sin 2x e^{-x} dx \right)$$

We can use integration by parts to solve the right side

$$= \lim_{m \to \infty} \left(-(\sin x)^2 e^{-x} \Big|_0^m + \frac{1}{5} e^{-x} (2\cos(2x) + \sin(2x)) \Big|_0^m \right)$$

Taking the limit as m goes to infinity, we get $\frac{2}{5}$

Problem 2. [Discussion 207] Determine whether or not the integral is convergent or divergent. If it is convergent, evaluate it. If it is divergent, prove that it diverges.

$$\int_0^\infty (\sin x)^2 e^{x} dx$$

Solution: Let us prove that this thing diverges. We know that

$$\int_0^\infty (\sin x)^2 e^{x} dx = \lim_{m \to \infty} \int_0^m (\sin x)^2 e^{x} dx$$

We know that $e^x > 1$ whenever $x > 0$

$$\geq \lim_{m \to \infty} \int_0^m (\sin x)^2 1 dx$$
$$\geq \lim_{m \to \infty} \int_0^m (\sin x)^2 dx$$
$$= \lim_{m \to \infty} \left(1/2(x - \cos(x)\sin(x)) \Big|_0^m \right)$$
$$= \lim_{m \to \infty} \left(1/2x \Big|_0^m \right) - \lim_{m \to \infty} \left(1/2 \cos(x)\sin(x) \Big|_0^m \right)$$

The term on the left goes to infinity, and the term on the right is always between 0 and 1, so the whole thing goes to infinity.

Problem 3. Find the surface area of the shape generated by rotating the curve $y^2 = x^3$ around the y axis, where x takes on values between 0 and 1.

Solution: We know that the surface area of a rotation around the y axis is given by

$$\int_a^b x ds$$

We have that $ds = \sqrt{1-(f'(x))^2}\, dx$. Computing $f'(x) = \frac{3}{2}x^{1/2}$,

$$\int_0^1 x\sqrt{1+(f'(x))^2}\, dx = \int_a^b x\sqrt{1-(\frac{3}{2}x^{1/2})^2}\, dx$$
$$= \int_0^1 x\sqrt{1+\frac{9}{4}x}\, dx$$

Substituting $u = 1 + \frac{9}{4}x$, and $du = \frac{9}{4}dx$

$$= \int_{u(0)}^{u(1)} \frac{4}{9}(u-1)\sqrt{u}\,\frac{4}{9}du$$
$$= \int_{u(0)}^{u(1)} \frac{16}{81}\left(u^{3/2} - u^{1/2}\right) du$$
$$= \frac{16}{81}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_{u(0)}^{u(1)}$$

If you got this far in the problem, you got full credit. Substituting back in $u(0) = 1$, and $u(1) = 1 + \frac{9}{4}$

$$= \frac{64 + 247\sqrt{13}}{1215}$$