1. Continuity Proof:

Last week on the homework, there was a problem that asked:

> If $f(x), g(x)$ and $q(x)$ are all polynomials, and you know that wherever $q(x) \neq 0$ you have
> $$\frac{f(x)}{q(x)} = \frac{g(x)}{q(x)}$$
> show that $f(x) = q(x)$

Here is a rigorous proof of this fact:

Proof. We will show that $f(x) = q(x)$ in two separate cases. For every x, we know that either $q(x) \neq 0$ or $q(x) = 0$.

- **Case 1:** Suppose that $q(x) \neq 0$. Then we have that
$$\begin{aligned} f(x) =& \frac{f(x)q(x)}{q(x)} \\ =& \frac{g(x)q(x)}{q(x)} \\ =& g(x) \end{aligned}$$
The second equality is a given assumption in the problem statement.
- **Case 2:** Suppose that $q(x) = 0$. As $q(x)$ is a polynomial, it has only a finite number of zeros. Therefore, we can find a sequence of points x_i with the following two properties:
  (1) $\lim_{i\to\infty} x_i = x$
  (2) $q(x_i) \neq 0$ for every x_i

  Aside: These two statements are a little tricky to think about. Convince yourself that this is true before proceeding through the rest of the proof. Reading proofs can be difficult the first time through, and it can take several minutes to even make it through a few sentences!

  With this sequence of x_i in hand, we can make the following equalities.

$$\begin{aligned} f(x) =& f\left(\lim_{i\to\infty} x_i\right) && (1) \\ =& \lim_{i\to\infty} f(x_i) && (2) \\ =& \lim_{i\to\infty} g(x_i) && (3) \\ =& g\left(\lim_{i\to\infty} x_i\right) && (4) \\ =& g(x) && (5) \end{aligned}$$

Therefore, whether or not $q(x) = 0$, we have that $f(x) = g(x)$. □

Notes: You should convince yourself that all of the equalities in the last calculation are true. In particular,

(1) Why in line 1 can we replace x with a limit?
(2) Why can we switch the limit and function in line 2
(3) *Especially Important*: Why is $f(x_i) = g(x_i)$ for all x_i?