Quiz 2

Name:

Problem 1. Find the integral of

$$\int \frac{e^x}{e^{2x}+1} dx$$

Solution: Make the substitution $e^x = u$, so $e^x dx = du$ Then the integral becomes

$$\int \frac{1}{u^2+1} du$$

which integrates to $\tan^{-1}(u) + C$. Making a substitution back in yields

$$\tan^{-1}(e^x) + C$$

Problem 2. Find the quadratic polynomial that fits the function

$$f(x) = x \sin(\pi x)$$

at $x = 0$, $x = 1$ and $x = 1/2$. Then use this polynomial to approximate $\int_0^1 x \sin(\pi x)\, dx$.

Solution: We plug in the values at $x = 0$, $x = 1$ and $x = 1/2$.

$$f(0) = 0 \qquad f(1/2) = 1/2 \qquad f(1) = 0$$

Now we will try to find a quadratic polynomial that fits this these three points. Let us use $P(x) = Ax^2 + Bx + C$.

$$\begin{aligned} 0 = P(0) =& A(0)^2 + B(0) + C \\ \frac{1}{2} = P(1/2) =& A(1/2)^2 + B(1/2) + C \\ 0 = P(1) =& A(1)^2 + B(1) + C \end{aligned}$$

This gives us that $C = 0$, $\frac{A}{4} + \frac{B}{2} = \frac{1}{2}$ and $A + B = 0$. Solving this system of linear equations (or squinting) will yield that $A = -2, B = 2$, so

$$P(x) = -2x^2 + 2x$$

Integrating $P(x)$ from 0 to 1 gives us a final estimate

$$\begin{aligned} \int_0^1 P(x)\, dx =& \int_0^1 -2x^2 + 2x\, dx \\ =& \frac{-2x^3}{3} + \frac{2x^2}{2}\Big|_0^1 \\ =& \frac{1}{3} \end{aligned}$$

Problem 3. Compute the following integral

$$\int \frac{-6-15x+7x^2+10x^3}{-2x-3x^2+x^4}dx$$

This one takes a bit of set up. First, we have to factor the bottom. We notice that right away we can pull out an x.

$$x^4-3x^2-2x = x(x^3-3x-2)$$

At this point, you have to use your favorite method for factoring a cubic polynomial. If the polynomial has rational roots, then the polynomial will have roots at $\pm 1, \pm 2$. In this case, if you try $+2$, it works, as $2^3-3\cdot 2-2=8-6-2=0$. Now, you use polynomial long division (which I don't know how to type on a computer) to get

$$= x(x-2)(x+2x+1)$$

The right hand side factors even further as

$$= x(x-2)(x+1)^2$$

Now comes the fun part: solving for coefficients. You write out

$$\frac{A}{x}+\frac{B}{(x-2)}+\frac{C}{(x+1)}+\frac{D}{(x+1)^2}=\frac{-6-15x+7x^2+10x^3}{-2x-3x^2+x^4}$$

Cross multiplying and looking at the numerator gives us

$$(\star)\quad A(x-2)(x+1)^2+Bx(x+1)^2+Cx(x-2)(x+1)+Dx(x-2)=-6-15x+7x^2+10x^3$$

Wow, that looks scary. Let us try substituting in $x=0$ for $(\star)$

$$\begin{aligned} A(0-2)(0+1)+B(0)+C(0)+D(0) &= -6-15(0)+7(0)^2+10(0)^3 \\ -2A &= -6 \\ A &= 3 \end{aligned}$$

Not so bad. Now let us try substituting $x=2$ for $(\star)$

$$\begin{aligned} A(0)+B(2)(3)^2+C(0)+D(0) &= -6-15(2)+7(2)^2+10(2)^3 \\ 18B &= 72 \\ B &= 4 \end{aligned}$$

Two more to go. Let $x=-1$ for $(\star)$

$$\begin{aligned} A(0)+B(0)+C(0)+D(-1)(-1-2) &= -6-15(-1)+7(-1)^2+10(-1)^3 \\ 3D &= 6 \\ D &= 2 \end{aligned}$$

One last one. Now we will substitute $x=1$, but also in the values of A, B and D we have found so far.

$$\begin{aligned} A(1-2)(1+1)^2+B(1)(1+1)^2+C(1)(1-2)(1+1)+D(1)(1-2) &= -6-15(1)+7(1)^2+10(1)^3 \\ -4A+4B-2C-D &= -4 \\ -4(3)+4(4)-2C-2 &= -4 \\ C &= 3 \end{aligned}$$

Great. Now all we have to do is integrate

$$\begin{aligned} \int \frac{A}{x}+\frac{B}{(x-2)}+\frac{C}{(x+1)}+\frac{D}{(x+1)^2}dx &= \int \frac{3}{x}+\frac{4}{(x-2)}+\frac{3}{(x+1)}+\frac{2}{(x+1)^2}dx \\ &= 3\ln|x|+4\ln|x-2|+3\ln|x+1|+\frac{2}{-(x+1)^1}+C \end{aligned}$$