1. Partial Fraction Decomposition!

One thing that was always interesting to me was why do we almost always have a partial fraction decomposition for our problems. In general, the problem looks something like this: $\frac{f(x)}{g(x)}$. where f and g are polynomials, and the degree of f is less than the degree of g. We usually also assume that g factors into linear components, that is,

$$g(x) = (x - a_1)(x - a_2)(x - a_3) \dots (x - a_n)$$

Where a_i are the roots of the polynomial g, and n is the degree of g. Then we might ask ourselves– why are there almost always constants $B_1, B_2, B_3 \dots B_n$ so that we have the equality

$$\frac{f(x)}{g(x)} = \frac{B_1}{x - a_1} + \frac{B_2}{x - a_2} + \dots + \frac{B_n}{x - a_n}$$

Here is a little trick that can help you solve these problems a little bit faster, and provides an explicit formula for the B_i.

When we solve a Partial fraction decomposition problem, we usually cross multiply the denominators on our first step to finding eligible values for the B_i. This gives us a generally nasty looking equation:

$$\frac{f(x)}{g(x)} = \frac{B_1(x - a_2)(x - a_3) \dots (x - a_n) + B_2(x - a_1)(x - a_3) \dots (x - a_n) + \dots B_n(x - a_1)(x - a_2) \dots (x - a_{n-1})}{g(x)}$$

On the next step of the problem we set the numerators for these two fractions to be equal. At this point, the equation we have is still pretty nasty looking:

$$f(x) = B_1(x - a_2)(x - a_3) \dots (x - a_n) + B_2(x - a_1)(x - a_3) \dots (x - a_n) + \dots B_n(x - a_1)(x - a_2) \dots (x - a_{n-1})$$

Here, we have a little trick that we can do to simplify the problem immensely. What happens if we were to plug in a_1 for x? On the right side of the equation, all of the terms that have $(x - a_1)$ as an entry will disappear, leaving us with

$$f(a_1) = B_1(a_1 - a_2)(a_1 - a_3)(a_1 - a_4) \dots (a_1 - a_n)$$

Similarly for the other B_i we could plug in a_i into the really nasty equation to get the equalities

$$\begin{aligned} f(a_2) =& B_2(a_2 - a_1)(a_2 - a_3)(a_2 - a_4) \dots (a_2 - a_n) \\ f(a_2) =& B_3(a_3 - a_1)(a_3 - a_2)(a_3 - a_4) \dots (a_3 - a_n) \\ &\vdots \\ f(a_n) =& B_n(a_n - a_1)(a_n - a_2)(a_n - a_3) \dots (a_n - a_{n-1}) \end{aligned}$$

This gives us explicit formulas for the B_i, that is

$$B_i = \frac{f(a_i)}{(a_i - a_1)(a_i - a_2) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)}$$

Notice in the denominator the term $(a_i - a_i)$ is missing. It would make no sense for that term to be there, because that would mean dividing by 0!

Let us do a quick example to show that this method works. Let us try decompose

$$\frac{4x + 1}{(x - 2)(x - 3)(x - 4)}.$$

Using the method above, it should decompose as

$$\frac{B_1}{x - 2} + \frac{B_2}{x - 3} + \frac{B_2}{x - 4}$$

Where

$$B_1 = \frac{4(2)+1}{(2-3)(2-4)} = \frac{9}{2}$$
$$B_2 = \frac{4(3)+1}{(3-2)(3-4)} = -13$$
$$B_2 = \frac{4(4)+1}{(4-2)(4-3)} = \frac{17}{2}$$