1. Integration by Parts

Bonus: You are given a bucket that ways 10 kg, and it is filled with 20l of water. You are carrying it up Evans hall (50m), and it takes you 5 minutes to complete this task. During this time, the bucket is leaking at a rate of 2 liters per minute. How much work is required to carry the bucket up the stairs?

(1)

$$\int \tan^{-1} x dx$$

Solution: Let $u = \tan^{-1}(x)$, and $v = x$. The integration by parts gives us a

$$x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

Making the substitution $u = x^2$ and $du = 2xdx$ we arrive at

$$x \tan^{-1} x - \int \frac{du}{2(1+u)} du$$

$$x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$$

(2) Compute the integral of $\int e^x \sin x dx$

Solution: . Integrate by parts. Let $u = \sin x$, and $dv = e^x dx$. Then

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

We repeat integration by parts on the second term, letting $u = \cos x$ and $dv = e^x dx$

$$= e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) dx \right) + C$$

Rearranging terms gives us

$$2 \int e^x \sin x dx = e^x \sin x - e^x \cos x + C$$

$$\int e^x \sin x dx = \frac{e^x \sin x - e^x}{2} + C$$

A different way to do this problem is with complex numbers. We know that $e^{ix} = \cos x + i \sin x$. This means that

$$\sin x = i \left(\frac{e^{ix} - e^{-ix}}{2} \right)$$

We can therefore rewrite our integral as

$$\begin{aligned}\int e^x \sin x dx = &\int ie^x \left(\frac{e^{ix} - e^{-ix}}{2}\right) dx \\ =&\frac{i}{2}\int e^x e^{ix} - e^x e^{-ix} dx \\ =&\frac{i}{2}\int e^{(1+i)x} - e^{(1-i)x} dx \\ =&\frac{i}{2}\left(\frac{e^{(1+i)x}}{1+i} - \frac{e^{(1-i)x}}{1-i}\right) \\ =&\frac{i}{2}\left((1/2 - i/2)e^{(1+i)x} - (1/2 + i/2)e^{(1-i)x}\right) dx \\ =&\frac{ie^x}{2}\left((1/2 - i/2)e^{ix} - (1/2 + i/2)e^{-ix}\right) = \qquad \frac{e^x}{2}(\sin x - \cos x)\end{aligned}$$

(3) Compute the indefinite integral of $\int \sin(\ln(x))dx$
Solution: Let $x = e^u$, so $u = \ln x$ and $e^u du = dx$. If we substitute this in the integral becomes

$$\int \sin(\ln x)dx = \int e^u \sin u du$$

This is exactly the previous problem, which we know how to solve.

(4) if f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy$$

Solution:

$$\int_a^b f(x)dx = \int_a^b 1 \cdot f(x)dx$$

Letting $u = f(x)$ and $dv = dx$

$$=bf(b) - af(a) = \int_a^b xf'(x)dx$$

Using that $x = g(f(x))$

$$=bf(b) - af(a) = \int_a^b g(f(x))f'(x)dx$$

Substituting $y = f(x)$ and $dy = f'(x)dx$

$$=bf(b) - af(a) \int_{f(a)}^{f(b)} g(y)dy$$