MATH N55 HOMEWORK 5
DUE FRIDAY, JULY 12TH

Do the following problems in Rosen.

Section 4.4: 6, 8, 10, 12, 13, 16, 26, 34

Challenge (Not to be handed in)

Find a polynomial $f(x)$ with integer coefficients which has roots in $\mathbb{Z}/p$ for all primes p, but which does not have any roots in $\mathbb{Z}$. More precisely, find an $f(x)$ such that for all p, there exists an $a \in \mathbb{Z}$ such that $f(a) \equiv 0 \pmod{p}$, does there does not exist any $a \in \mathbb{Z}$ such that $f(a) = 0$. *Possible hint*: you may use the results from exercises 61–63 in section 4.4 without proving them.

Going Further. Borrowing language from geometry, number theorists would say that $f(x)$ has roots "locally," but not "globally." Theorems which relate solutions in $\mathbb{Z}$ or $\mathbb{Q}$ to solutions modulo each p are called "local-to-global principals."

The language comes from a deep analogy with geometry, where prime numbers correspond to "points" and integers correspond to "functions." As a familiar starting point: every nonzero polynomial function $g(x)$ on $\mathbb{C}$ factors uniquely into terms of the form $x - a$, for points $a \in \mathbb{C}$ where $g(a) = 0$. Every nonzero integer n factors uniquely into primes p, where $n \equiv 0 \pmod{p}$. The full story involves the closely related subjects of *commutative algebra* and *algebraic geometry*.