SOLUTIONS FOR MATH 1B WEEK 8, TUESDAY

Exercise 1. Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{((n+1)!)^2/(2n+2)!}{(n!)^2/(2n)!}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} < 1$$

Thus the series converges.

Exercise 2. Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{2^{n+1}/(n+1)^{1{,}000{,}000}}{2^n n^{1{,}000{,}000}}\right| = \lim_{n\to\infty} 2\left(\frac{n+1}{n}\right)^{1{,}000{,}000} = 2$$

$\sum_{n=1}^{\infty}\frac{2^n}{n^{1{,}000{,}000}}$ Thus the series converges.

★ **Exercise 3.** Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(n+1)^{n+1}/(n+1)!}{n^n/n!}\right| = \lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1.$$

Thus the series diverges.

Exercise 4. This can be re-written as a geometric series in $\frac{x}{4}$. Thus it converges iff $|x/4| < 1$, or equivalently, $|x| < 4$. This corresponds to the interval $(-4, 4)$.

Exercise 5. Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{((n+1)^3+1)x^{n+1}}{(n^3+1)x^n}\right| = |x|.$$

Thus if $|x| < 1$, we have convergence and if $|x| > 1$, we have divergence. At $|x| = 1$, we check by plugging in:

$$\sum_{n=0}^{\infty}(n^3+1)(1)^n, \qquad \sum_{n=0}^{\infty}(n^3+1)(-1)^n$$

Both of these series diverge since the general term does not tend to zero. Thus the series converges on $(-1, 1)$.

★ **Exercise 6.** Intuitively, this looks like $\frac{n^2}{n^x} = \frac{1}{n^{x-2}}$. Indeed, we have

$$\lim_{n\to\infty}\frac{(n^2+1)/n^x}{1/n^{x-2}} = \lim_{n\to\infty}\frac{n^2+1}{n^2} = 1$$

So by the limit comparison test, our series converges iff $\sum\frac{1}{n^{x-2}}$ does. By the p-series test, this converges iff $x - 2 > 1$, or equivalently if $x > 3$. Thus this series converges on $(3, \infty)$.

Remark: Series of the form $\sum\frac{a_n}{n^x}$ are called *Dirichlet series*. The most famous example of such a series is the *Riemann zeta function* $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$.