SOLUTIONS FOR MATH 1B WEEK 8, THURSDAY

Exercise 1.

$$\frac{x}{x+2} = x \cdot \frac{1}{2} \cdot \frac{1}{1 - \left(-\frac{x}{2}\right)} = x \cdot \frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^m} x^m$$

This is valid for $\left|\frac{x}{2}\right| < 1$, or equivalently $|x| < 2$.

Exercise 2. Since $\frac{d}{dx} \ln(x+1) = \frac{1}{x+1}$ and $\ln(1) = 0$, we have

$$\begin{aligned}
\ln(x+1) &= \int_0^x \frac{1}{t+1} dt \\
&= \int_0^x \sum_{n=0}^{\infty} (-t)^n dt \\
&= \sum_{n=0}^{\infty} (-1)^n \int_{n=0}^{\infty} t^n dt \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} x^m
\end{aligned}$$

Exercise 3. The series we want is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, which is exactly the series above evaluated at $x = 1$. Thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(1+1) = \ln 2$$

Exercise 4. We have $f(0) = a_0$, so condition (2) forces us to pick $a_0 = 1$. For the rest, recall what a derivative does to the coefficients of a power series:

$$\frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

We want to find values of a_n which makes this again equal to the original series, to satisfy condition (1). That is, we need

$$\begin{array}{ccc}
a_1 = a_0 & \Longrightarrow & a_1 = 1 \\
2a_2 = a_1 & \Longrightarrow & a_2 = \dfrac{1}{2} \\
3a_3 = a_2 & \Longrightarrow & a_3 = \dfrac{1}{2 \cdot 3} \\
4a_4 = a_3 & \Longrightarrow & a_4 = \dfrac{1}{2 \cdot 3 \cdot 4}
\end{array}$$

Continuing in this pattern leads us to guess $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$. Using the ratio test, one can verify that this converges for all x, so this is a well-defined function. We have $f(0) = \frac{1}{0!} = 1$, and further,

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \cdot nx^{n-1} = \sum_{n=1} \frac{1}{(n-1)!}x^n = \sum_{m=0} \frac{1}{m!}x^m = f(x).$$

Thus, f satisfies both conditions.

Remark: If you haven't seen a satisfying definition of what a^b should mean for irrational b, or of what the number e is, you can take this series to be the *definition* of the function e^x. The fact that this function has the familiar properties of exponentiation, and has an inverse, $\ln x$, can be proved from (1) and (2) alone (try it if you're bored!).