SOLUTIONS FOR MATH 1B WEEK 7, TUESDAY

Exercise 1. Re-write the sum as

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}.$$

Then since $\frac{1}{2n+1}$ is positive, decreasing, and tends to 0, we can apply the alternating series test to show this converges.

Exercise 2. Ignoring everything but the leading terms, this looks like $\frac{1}{n^2}$, which we know converges. Make this precise by using the limit comparison test:

$$\lim_{n\to\infty} \frac{1/(n+1)(n-1)}{1/n^2} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{n}{n-1} = 1.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)}$.

Exercise 3. For small values of x, recall that $\sin x \approx x$, and so for large values of n, $\sin\left(\frac{1}{n}\right) \approx \frac{1}{n}$. Make this precise using the limit comparison test:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, neither does $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$.

Exercise 4. This looks like an alternating series, but if you try to apply the alternating series test it will fail, because $\frac{1+\cos(\pi n)}{n}$ oscillates, so it is not eventually decreasing. Thus we need to use other methods to determine if this series converges.

Writing out the first few terms gives

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1+\cos(\pi n)}{n} = 0 - 1 + 0 - \frac{1}{2} + 0 - \frac{1}{3} + 0 - \frac{1}{4} + \cdots$$

so it looks like every odd term is 0 and every even term is $-\frac{2}{n}$. This holds for more generally for n, since $\cos(\pi n)$ is -1 for n odd and 1 for n even. It is clear that the $2N$-th partial sum of this series is the negative of the N-th partial sum of the harmonic series. We know the partial sums of the harmonic series tend to ∞, so the partial sums of this series tend to $-\infty$, meaning it diverges.

Exercise 5. (*Remark:* The solution to this exercise, suitably generalized, is why the ratio test works. If you just are concerned with doing this exercise quickly, e.g. on an exam, just use the ratio test like on week 8's worksheet.) We have

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = \frac{1}{1} + \frac{2}{1} + \frac{2 \cdot 2}{1 \cdot 2} + \frac{2 \cdot 2 \cdot 2}{1 \cdot 2 \cdot 3} + \frac{2 \cdot 2 \cdot 2 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots + \frac{\overbrace{2 \cdots 2}^{n \text{ times}}}{1 \cdots n} + \cdots$$

Notice that each term is the previous, multiplied by $\frac{2}{n}$. This is eventually < 1. More precisely, for $n \geq 3$, we have $\frac{2}{n} \leq \frac{2}{3} < 1$, and so

$$a_n = \frac{2 \cdot 2 \cdot \overbrace{2 \cdot 2 \cdots 2}^{n-2 \text{ times}}}{1 \cdot 2 \cdot 3 \cdot 4 \cdots n} \leq \frac{2 \cdot 2}{1 \cdot 2} \left(\frac{2}{3}\right)^{n-2}$$

Since $\sum_{n=0}^{\infty} 2 \left(\frac{2}{3}\right)^{n-2}$ is a geometric series with ratio < 1, it converges. By the comparison test, our original series converges as well.

Exercise 6. We want to apply the alternating series test. To do this, we need to show

$$a_n := \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!}$$

is positive, decreasing, and tends to 0. That it is positive is clear. To show it is decreasing, notice that each term is $\frac{2n-3}{2n}$ times the previous. Since $\frac{2n-3}{2n} < \frac{2n}{2n} = 1$, each term is strictly smaller than the previous. To show it tends to zero, pair up terms:

$$a_n = \frac{\overbrace{1 \cdot 3 \cdot 5 \cdots (2n-3)}^{n-1 \text{ factors}}}{\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ factors}} \cdot \underbrace{1 \cdot 2 \cdot 3 \cdots n}_{n \text{ factors}}} = \overbrace{\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-3}{2n-2}}^{n-1 \text{ factors, all } <1} \cdot \frac{1}{2n} \leq \frac{1}{2n} \to 0.$$

Since all the hypotheses of the alternating series test are satisfied, this series converges.