SOLUTIONS FROM MATH 1B WEEK 4, TUESDAY

Exercise 1.

$$\begin{aligned}\int_2^\infty e^{-5p}dp &= \lim_{a\to\infty}\int_2^a e^{-5p}dp \\ &= \lim_{a\to\infty} -\frac{1}{5}(e^{-5a}-e^{-10}) \\ &= \frac{1}{5}e^{-10}\end{aligned}$$

Exercise 2. One could compute this limit out using a trig substitution. However, right away we can notice that this should diverge. Intuitively, as x gets very large, $\sqrt{1+x^2}$ behaves like $\sqrt{x^2}=x$, and so the integrand behaves like $x^2/x=x$. Since we know $\int_0^\infty xdx$ should diverge, so should our integral.

To make this precise, we use a comparison test. It is sufficient to show that $\int_1^\infty \frac{x^2}{\sqrt{1+x^2}}dx$ diverges. (Note the different bound.) For $x\geq 1$, we have $x^2\geq 1$, and so $1+x^2\leq x^2+x^2=2x^2$. Thus

$$\frac{x^2}{\sqrt{1+x^2}}\geq\frac{x^2}{\sqrt{2x^2}}=\frac{x}{\sqrt{2}}$$

Since $\int_1^\infty \frac{x}{\sqrt{2}}dx$ diverges, so does our original integral.

Exercise 3.

$$\begin{aligned}\int_{-\infty}^\infty xe^{-x^2}dx &= \lim_{a\to-\infty}\int_a^0 xe^{-x^2}dx+\lim_{a\to\infty}\int_0^a xe^{-x^2}dx \\ &= \lim_{a\to-\infty}-\frac{1}{2}(e^0-e^{-a^2})+\lim_{a\to\infty}-\frac{1}{2}(e^{-a^2}-e^0) \\ &= -\frac{1}{2}+\frac{1}{2}=0\end{aligned}$$

Exercise 4.

$$\begin{aligned}\int_1^\infty \frac{1}{x^2+x}dx &= \lim_{a\to\infty}\int_1^a \frac{1}{x^2+x}dx \\ &= \lim_{a\to\infty}\int_1^a\left(\frac{1}{x}-\frac{1}{x+1}\right)dx \\ &= \lim_{a\to\infty}(\ln a-\ln(a+1)-\ln 1+\ln 2) \\ &= \lim_{a\to\infty}\ln\left(\frac{2a}{a+1}\right) \\ &= \ln 2\end{aligned}$$

Exercise 5.

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x}dx &= \lim_{a\to\infty}\int_1^a \frac{\ln x}{x}dx \\ &= \lim_{a\to\infty}(\ln a)^2-(\ln 1)^2=\infty\end{aligned}$$

So the integral diverges.

Alternatively, notice that for $x \geq e$, we have $\ln x \geq 1$ and so $\frac{\ln x}{x} \geq \frac{1}{x}$. Since we already know $\int_1^\infty \frac{1}{x}dx$ diverges, this integral must diverge.

Exercise 6.

$$\begin{aligned}\int_0^1 \frac{1}{x}dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{x}dx \\ &= \lim_{a\to 0^+}(\ln 1-\ln a)=\infty\end{aligned}$$

So the integral diverges.