SOLUTIONS FOR MATH 1B WEEK 3, THURSDAY

Exercise 1. Let $u = 1 - \sin x$, so $du = -\cos x$. Then

$$\int \frac{\cos x}{1 - \sin x} dx = -\int \frac{1}{u} du = \ln|u| + C = \ln(1 - \sin x) + C$$

Exercise 2. Using a clever substitution, we can reduce this to $\int \ln u du$, which we have already seen how to do. The idea is that ln "eats powers." Let $u = y^{3/2}$, so that $du = \frac{3}{2}\sqrt{y}dy$. We have $y = u^{2/3}$, so that

$$\int \sqrt{y} \ln y dy = \frac{2}{3} \int \ln(u^{2/3}) du$$
$$= \frac{4}{9} \int \ln u du$$

Using a previous computation for $\int \ln u du$, we know this is

$$\frac{4}{9} u(\ln u - 1) + C = \frac{4}{9} y^{3/2}(\ln(y^{3/2}) - 1) + C$$

Exercise 3. We see a linear term in the numerator, which hints to us we should make a quadratic substitution. Let $u = t^2$, so that $du = 2tdt$. Then

$$\int \frac{t}{t^4 + 2} dt = \frac{1}{2} \int \frac{1}{u^2 + 2} du$$

Now we can solve this with a trig sub, letting $u = \sqrt{2} \tan\theta$ and $du = \sqrt{2} \sec^2 \theta d\theta$. This gives

$$\frac{1}{2} \int \frac{\sqrt{2} \sec^2 \theta d\theta}{2 \tan^2 \theta + 2} = \frac{1}{2\sqrt{2}} \int d\theta$$
$$= \frac{1}{2\sqrt{2}} \theta + C$$
$$= \frac{1}{2\sqrt{2}} \arctan\left(\frac{u}{\sqrt{2}}\right) + C$$
$$= \frac{1}{2\sqrt{2}} \arctan\left(\frac{t^2}{\sqrt{2}}\right) + C$$

Exercise 4. Let $u = \arctan y$, so that $du = \frac{1}{1 + y^2} dy$, and thus

$$\int \frac{e^{\arctan y}}{1 + y^2} dy = \int e^u du = e^u + C = e^{\arctan y} + C$$

Exercise 5. Factor the denominator as $(x+4)(x-1)$, and use partial fractions. We want to solve

$$\frac{A}{x+4} + \frac{B}{x-1} = \frac{x+2}{(x+4)(x-1)}$$

This gives $A(x-1) + B(x+4) = x+2$, or $A + B = 1$ and $-A + 4B = 2$. This is solved by $B = 3/5$ and $A = 2/5$. Thus our integral becomes

$$\int \left(\frac{3}{5} \frac{1}{x+4} + \frac{2}{5} \frac{1}{x-1} \right) dx = \frac{3}{5} \ln|x+4| + \frac{2}{5} \ln|x-1| + C$$

Exercise 6. We could use partial fractions directly here, but there is a quicker way. Let $u = x^3$, so that $du = 3x^2$. Then

$$\begin{aligned} \int \frac{x^2}{x^6 + 3x^3 + 2} dx &= \frac{1}{3} \int \frac{du}{u^2 + 3u + 2} \\ &= \frac{1}{3} \int \left(\frac{1}{u+1} - \frac{1}{u+2} \right) du \\ &= \frac{1}{3} (\ln|u+1| - \ln|u+2|) + C \\ &= \frac{1}{3} \left(\ln|x^3+1| - \ln|x^3+2| \right) + C \end{aligned}$$