SOLUTIONS FOR MATH 1B WEEK 3, TUESDAY

Exercise 1. Let $x = 3\sin\theta$ for $\theta \in (-\pi/2, \pi/2)$. Then $dx = 3\cos\theta d\theta$.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-x^2}}dx &= \int \frac{27\sin^2\theta\cos\theta}{\sqrt{9-9\sin^2\theta}}d\theta \\
&= \int \frac{9\sin^2\theta\cos\theta}{|\cos\theta|}d\theta \\
&= \int 9\sin^2\theta d\theta \\
&= \int \left(\frac{9}{2} - \frac{9}{2}\cos(2\theta)\right) d\theta \\
&= \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C \\
&= \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + C \\
&= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{3-x^2} + C
\end{aligned}$$

Exercise 2. Let $x = a\tan\theta$, for $\theta \in (-\pi/2, \pi/2)$. Then $dx = a\sec^2\theta d\theta$. Further, let $u = \sin\theta$.

$$\begin{aligned}
\int_0^a \frac{dx}{(a^2+x^2)^{3/2}} &= \int_0^{\pi/4} \frac{1}{(a^2+a^2\tan^2\theta)^{3/2}}dx \\
&= \int_0^{\pi/4} \frac{1}{a^3\sec^3\theta}d\theta \\
&= \int_0^{\pi/4} \frac{1}{a^3}\cos^3\theta d\theta \\
&= \int_0^{\sqrt{2}/2} \frac{1}{a^3}(1-u^2)du \\
&= \frac{1}{a^3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12}\right)
\end{aligned}$$

Exercise 3. The polynomial in the denominator has discriminant $2^2 - 4 \cdot 5 = -16$. Since this is negative, we will not be able to factor it, so we should complete the square. Let $u = x+1$, so $du = dx$ and

$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{dx}{\sqrt{(x+1)^2+4}} = \int \frac{du}{\sqrt{u^2+4}}$$

Now do a trig sub with $u = 2\tan\theta$, so $du = 2\sec^2\theta d\theta$. Use the domain $\theta \in (-\pi/2, \pi/2)$, so that $\tan\theta$ ranges over all real numbers and $\sec\theta$ is positive. We get

$$\begin{aligned}\int \frac{2\sec^2\theta d\theta}{\sqrt{4\tan^2\theta + 4}} &= \int \frac{2\sec^2\theta d\theta}{2\sec\theta} \\ &= \int \sec\theta d\theta \\ &= \ln|\sec\theta + \tan\theta| + C\end{aligned}$$

To substitute back into u and then x, use the following two facts: (Note we don't run into issues with the square root on the second one since $\sec\theta$ is positive on $(-\pi/2, \pi/2)$.)

$$\tan\theta = \frac{1}{2}u = \frac{1}{2}(x+1)$$

$$\sec\theta = \sqrt{1 - \tan^2\theta} = \sqrt{1 - \frac{1}{4}u^2} = \sqrt{1 - \frac{1}{4}(x+1)^2}$$

This gives

$$\ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{1}{2}(x+1) + \sqrt{1 - \frac{1}{4}(x+1)^2}\right| + C.$$

Exercise 4. This one should have been $\int x\sqrt{1-x^4}dx$. In that case, it would be easy to substitute $u = x^2$ so the problem becomes $\frac{1}{2}\int \sqrt{1-u^2}du$ and then use a trig sub. Without it, this is actually a very tricky integral and can't be solved using methods we've used so far.

Exercise 5. Set $u = x^2 - 7$, so $du = 2xdx$. Then

$$\begin{aligned}\int \frac{x}{\sqrt{x^2-7}}dx &= \frac{1}{2}\int \frac{1}{\sqrt{u}}du \\ &= \sqrt{u} + C \\ &= \sqrt{x^2-7} + C\end{aligned}$$

Always remember to try easier methods, like u-substitution, before more time-consuming ones!

Exercise 6. Let $x = \tan\theta$, and use the domain $\theta \in (-\pi/2, \pi/2)$. Then $dx = \sec^2\theta d\theta$. Since $\tan 0 = 0$ and $\tan(\pi/4) = 1$, We have

$$\begin{aligned}
\int_0^1 \frac{dx}{(x^2+1)^2} &= \int_0^{\pi/4} \frac{\sec^2\theta d\theta}{(\tan^2\theta+1)^2} \\
&= \int_0^{\pi/4} \frac{\sec^2\theta d\theta}{\sec^4\theta} \\
&= \int_0^{\pi/4} \cos^2\theta d\theta \\
&= \frac{1}{2}\int_0^{\pi/4} (1+\cos(2\theta))d\theta \\
&= \frac{1}{2}\left(x + \frac{1}{2}\sin(2\theta)\right)\Bigg|_0^{\pi/4} \\
&= \frac{\pi}{8} + \frac{1}{4}
\end{aligned}$$

Exercise 7. Let $t = 2\tan\theta$, so $dt = 2\sec^2\theta d\theta$. Use the domain $\theta \in (-\pi/2, \pi/2)$. On this interval, note that $\sec\theta$ is positive. Since $2\tan 0 = 0$ and $2\tan(\pi/4) = 2$, we have

$$\begin{aligned}
\int_0^2 \frac{dt}{\sqrt{4+t^2}} &= \int_0^{\pi/4} \frac{2\sec^2\theta d\theta}{\sqrt{4+4\tan^2\theta}} \\
&= \int_0^{\pi/4} \frac{2\sec^2\theta d\theta}{\sec\theta} \\
&= \int_0^{\pi/4} \sec\theta d\theta \\
&= \ln|\sec\theta + \tan\theta|\Bigg|_0^{\pi/4} \\
&= \ln(\sqrt{2}+1) - \ln 1 = \ln(\sqrt{2}+1)
\end{aligned}$$