SOLUTIONS FROM MATH 1B WEEK 2, THURSDAY

Exercise 1. Let $u = \cos\theta$, so $du = -\sin\theta d\theta$. Then

$$\begin{aligned}
\int \sin^3\theta\cos^4\theta &= \int \sin\theta(1-\cos^2\theta)\cos^4\theta d\theta \\
&= -\int (1-u^2)u^4 du \\
&= -\int u^4 - u^6 du \\
&= -\frac{1}{5}u^5 + \frac{1}{7}u^7 + C \\
&= -\frac{1}{5}\cos^5\theta + \frac{1}{7}\cos^7\theta + C
\end{aligned}$$

Exercise 2. Let $u = \cos x$, so $du = -\sin x dx$. Also, $\cos 0 = 1$ and $\cos\pi/2 = 1$. Then

$$\begin{aligned}
\int_0^{\pi/2} \sin^5 x dx &= \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x dx \\
&= -\int_0^1 (1-u^2)^2 du \\
&= \int_0^1 (-1+2u^2-u^4) du \\
&= -1 + \frac{2}{3} - \frac{1}{5} = -\frac{8}{15}
\end{aligned}$$

Exercise 3. Let $u = t^2$, so $du = 2tdt$. Thus

$$\int t\cos^5(t^2)dt = \frac{1}{2}\int \cos^5(u)du$$

Now let $v = \sin u$, so $dv = \cos u du$.

$$\begin{aligned}
\frac{1}{2}\int \cos^5 u du &= \frac{1}{2}\int (1-\sin^2 u)^2 \cos u du \\
&= \frac{1}{2}\int (1-v^2)^2 dv \\
&= \frac{1}{2}\int (1-2v^2+v^4) dv \\
&= \frac{1}{2}v - \frac{1}{3}v^3 + \frac{1}{10}v^5 + C \\
&= \frac{1}{2}\sin(t^2) - \frac{1}{3}\sin^3(t^2) + \frac{1}{10}\sin^5(t^2) + C
\end{aligned}$$

Exercise 4.

$$\int_0^{2\pi} \sin^2(\frac{1}{3}\theta)d\theta = \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2}{3}\theta\right)\right) dx$$
$$= \pi - \frac{3}{4}\sin(4\pi/3) = \pi + \frac{3\sqrt{3}}{8}$$

Exercise 5. Let $u = \sec x$, so $du = \sec x \tan x dx$.

$$\int \tan^3 x \sec^6 x dx = \int (\sec^2 x - 1)\sec^5 x \cdot \sec x \tan x dx$$
$$= \int (u^2 - 1)u^5 du$$
$$= \int u^7 - u^5 du$$
$$= \frac{1}{8}u^8 - \frac{1}{6}u^6 + C$$
$$= \frac{1}{8}\sec^8 x - \frac{1}{6}\sec^6 x + C$$

Exercise 6. Use the identity $\tan^2 = 1 - \sec^2 x$ repeatedly, so that

$$\int \tan^5 x dx = \int (\tan^3 x - \sec^2 x \tan^3 x)dx$$
$$= \int (\tan x - \sec^2 \tan x - \sec^2 \tan^3 x)dx$$
$$= -\ln|\cos x| - \frac{1}{2}\tan^2 x - \frac{1}{4}\tan^4 x + C$$

Exercise 7. First, consider the easier integral $\int \sin^3 x dx$. Let $u = \cos x$ so $du = -\sin x dx$.

$$\int \sin^3 x = -\int (1 - u^2)dx = -u + \frac{1}{3}u^3 + C$$
$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

Now we solve the original integral with integration by parts. Let

$$u = x \qquad du = dx$$
$$v = -\cos x + \frac{1}{3}\cos^3 x \qquad dv = \sin^3 x dx$$

So that

$$\begin{aligned}\int x\sin^3 x dx &= x\left(-\cos x + \frac{1}{3}\cos^3 x\right) - \int \left(-\cos x + \frac{1}{3}\cos^3 x\right) dx \\ &= x\left(-\cos x + \frac{1}{3}\cos^3 x\right) + \sin x + \frac{1}{3}\int (1-\sin^2 x)\cos x dx \\ &= x\left(-\cos x + \frac{1}{3}\cos^3 x\right) + \sin x + \frac{1}{3}\int (1-w^2) dw \\ &= x\left(-\cos x + \frac{1}{3}\cos^3 x\right) + \sin x + \frac{1}{3}\sin x - \frac{1}{9}\sin^3 x + C\end{aligned}$$