SOLUTIONS FOR MATH 1B WEEK 2, TUESDAY

Exercise 1. Integrate by parts. Let

$$u = z \qquad du = dz$$
$$v = -\frac{1}{\ln 10}10^{-z} \qquad dv = 10^{-z}dz$$

so that

$$\begin{aligned}\int \frac{z}{10^z}dz &= -\frac{1}{\ln 10}z10^{-z} + \frac{1}{\ln 10}\int 10^{-z}dz \\ &= -\frac{1}{\ln 10}z10^{-z} - \frac{1}{(\ln 10)^2}10^{-z} + C\end{aligned}$$

Exercise 2. Integrate by parts. Let

$$u = \cos(2\theta) \qquad du = -2\sin(2\theta)d\theta$$
$$v = -e^{-\theta} \qquad dv = e^{-\theta}d\theta$$

Then

$$\int e^{-\theta}\cos(2\theta)d\theta = -e^{-\theta}\cos(2\theta) - 2\int e^{-\theta}\sin(2\theta)d\theta$$

Integrate by parts again, this time with

$$u = \sin(2\theta) \qquad du = 2\cos(2\theta)$$
$$v = -e^{-\theta} \qquad dv = e^{-\theta}d\theta$$

So that

$$\begin{aligned}\int e^{-\theta}\cos(2\theta)d\theta &= -e^{-\theta}\cos(2\theta) - 2\left(-e^{-\theta}\sin(2\theta) + 2\int e^{-\theta}\cos(2\theta)d\theta\right) \\ 5\int e^{-\theta}\cos(2\theta)d\theta &= 2e^{-\theta}\sin(2\theta) - e^{-\theta}\cos(2\theta) \\ \int e^{-\theta}\cos(2\theta)d\theta &= \frac{2}{5}2e^{-\theta}\sin(2\theta) - \frac{1}{5}e^{-\theta}\cos(2\theta) + C\end{aligned}$$

Exercise 3. Integrate by parts twice, picking the polynomial for u each time:

$$\begin{aligned}\int_0^1 (x^2+1)e^{-x}dx &= -(x^2+1)e^{-x} + 2\int xe^{-x}dx \\ &= -(x^2+1)e^{-x} + 2\left(-xe^{-x} + \int e^{-x}dx\right) \\ &= -(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x} + C\end{aligned}$$

Exercise 4. Use polynomial long division to write $x^4 = (x^3 + x^2 + x + 1)(x - 1) + 1$. Thus

$$\begin{aligned}\int \frac{x^4}{x-1}dx &= \int (x^3 + x^2 + x + 1)dx + \int \frac{1}{x-1}dx \\ &= \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln|x-1| + C\end{aligned}$$

Exercise 5. Factor the denominator as $(2x + 1)(x + 1)$. Then we wish to solve

$$\begin{aligned}\frac{A}{2x+1} + \frac{B}{x+1} &= \frac{2}{(2x+1)(x+1)} \\ A(x+1) + B(2x+1) &= 2 \\ (A+2B)x + (A+B) &= 2\end{aligned}$$

We do this by solving the linear system

$$\begin{aligned}A + 2B &= 0 \\ A + B &= 2\end{aligned}$$

This has solution $A = 4$, $B = -2$. Thus

$$\begin{aligned}\int_0^1 \frac{2}{2x^2+3x+1}dx &= \int_0^1 \left(\frac{4}{2x+1} - \frac{2}{x+1}\right) dx \\ &= 2\ln|2x+1| - 2\ln|x+1|\Big|_0^1 \\ &= 2(\ln 3 - \ln 1 - \ln 2 + \ln 0) = \ln\frac{9}{4}\end{aligned}$$

Exercise 6. Factor the denominator as $(t - 1)^2(t + 1)^2$. We wish to solve

$$\begin{aligned}\frac{A}{(t-1)^2} + \frac{B}{t-1} + \frac{C}{(t+1)^2} + \frac{D}{t+1} &= \frac{1}{(t-1)^2(t+1)^2} \\ A(t+1)^2 + B(t-1)(t+1)^2 + C(t-1)^2 + D(t-1)^2(t+1) &= 1 \\ (B+D)t^3 + (A+B+C-D)t^2 + (2A-B-2C-D)t + (A-B+C+D) &= 1\end{aligned}$$

This gives the linear system

$$\begin{aligned}B + D &= 0 \\ A + B + C - D &= 0 \\ 2A - B - 2C - D &= 0 \\ A - B + C + D &= 1\end{aligned}$$

which has solution $A = C = D = 1/4$, $B = -1/4$. Thus

$$\begin{aligned}\int \frac{dt}{(t^2-1)^2} &= \int \left(\frac{1}{4(t-1)^2} - \frac{1}{4(t-1)} + \frac{1}{4(t+1)^2} + \frac{1}{4(t+1)}\right) dt \\ &= -\frac{1}{4(t-1)} - \frac{1}{4}\ln|t-1| - \frac{1}{4(t+1)} + \frac{1}{4}\ln|t+1| + C\end{aligned}$$

Exercise 7. We need to solve

$$\begin{aligned}\frac{A}{x-1}+\frac{Bx+C}{x^2+9}&=\frac{10}{(x-1)(x^2+9)}\\ A(x^2+9)+(Bx+C)(x-1)&=10\\ (A+B)x^2+(-B+C)x+(9A-C)&=10\end{aligned}$$

This gives the linear system

$$\begin{aligned}A+B&=0\\ -B+C&=0\\ 9A-C&=10\end{aligned}$$

This has the solution $A=1$ and $B=C=-1$. Thus

$$\begin{aligned}\int\frac{10}{(x-1)(x^2+9)}dx&=\int\left(\frac{1}{x-1}-\frac{x+1}{x^2+9}\right)dx\\ &=\int\left(\frac{1}{x-1}-\frac{x}{x^2+9}-\frac{1}{x^2+9}\right)dx\\ &=\ln|x-1|-\frac{1}{2}\ln(x^2+9)-\frac{1}{3}\arctan\left(\frac{x}{3}\right)+C\end{aligned}$$

Exercise 8. Start by completing the square and substituting $u=x+1$:

$$\begin{aligned}\int\frac{x+4}{x^2+2x+5}dx&=\int\frac{x+4}{(x+1)^2+4}dx\\ &=\int\frac{u+3}{u^2+4}du\\ &=\int\left(\frac{u}{u^2+4}+\frac{3}{u^2+4}\right)du\\ &=\frac{1}{2}\ln(u^2+4)+\frac{3}{2}\arctan\left(\frac{u}{2}\right)+C\\ &=\frac{1}{2}\ln((x+1)^2+4)+\frac{3}{2}\arctan\left(\frac{x+1}{2}\right)+C\end{aligned}$$

Exercise 9. Let $u=\cos x$, so $du=-\sin x dx$, and

$$\int\frac{\sin x}{\cos^2 x-3\cos x}dx=\int\frac{-1}{u^2-3u}du$$

Now we wish to solve

$$\begin{aligned}\frac{A}{u}+\frac{B}{u-3}&=\frac{-1}{u(u-3)}\\ A(u-3)+Bu&=-1\\ (A+B)u-3A&=-1\end{aligned}$$

So we have the linear system

$$\begin{aligned} A + B &= 0 \\ -3A &= -1 \end{aligned}$$

This has the solution $A = 1/3$ and $B = -1/3$. Thus

$$\begin{aligned} \int \frac{-1}{u(u-3)} dx &= \int \left(\frac{1}{3u} - \frac{1}{3(u-3)} \right) du \\ &= \frac{1}{3} \ln |u| - \frac{1}{3} \ln |u - 3| + C \\ &= \frac{1}{3} \ln |\cos x| - \frac{1}{3} \ln |\cos x - 3| + C \end{aligned}$$