MATH 1B WEEK 1 SOLUTIONS

Exercise 1. $\frac{d}{dx}(x^2 + \sin(2x)) = 2x + 2\cos x$

Exercise 2. $\frac{d}{dx}(\ln(2x)\tan(2^x)) = \frac{1}{x}\tan(2^x) + \ln(2x)\sec^2(2^x)2^x(\ln 2)$

Exercise 3. First compute $\frac{d}{dx}x^x$. Let $y = x^x$, so $\ln y = \ln(x^x) = x\ln x$. Differentiating both sides with respect to x gives $\frac{1}{y}\frac{dy}{dx} = \ln x + 1$. Thus $\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$. Applying this and using the chain rule, we get $\frac{d}{dx}\sec\left(e^{x^x}\right) = \sec\left(e^{x^x}\right)\tan\left(e^{x^x}\right)e^{x^x}x^x(\ln x + 1)$.

Exercise 4. $\lim_{x\to\infty}\frac{3x^2 + x - 1}{3x - 1} = \lim_{x\to\infty}\frac{3x + 1 - \frac{1}{x}}{3 - \frac{1}{x}}$. The denominator approaches 3 and the numerator approaches $+\infty$, so the limit is $+\infty$. (Compare with the "cross out all but the highest degree term" method, or with using L'Hôspital's rule.)

Exercise 5. $\lim_{x\to 0}x^x = \lim_{x\to 0}e^{x\ln x}$. We first compute $\lim_{x\to 0}x\ln x = \lim_{x\to 0}\frac{\ln x}{x^{-1}}$. By L'Hôspital's rule, this is $\lim_{x\to 0}\frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0}-x = 0$. Then since e^x is a continuous function of x, we have $\lim_{x\to 0}e^{x\ln x} = e^0 = 1$.

Exercise 6. Use u-substitution. Let $u = x^2 + 1$, so $du = 2xdx$. Then $\int\frac{2x}{x^2+1}dx = \int\frac{1}{u}du = \ln|u| + C = \ln|x^2 + 1| + C$.

Exercise 7. Use u-substitution. Let $u = 5x + 10$, so $x = \frac{1}{5}u - 2$ and $dx = \frac{1}{5}du$. Then

$$\begin{aligned}
\int x\sqrt{5x+10}dx &= \int\left(\frac{1}{5}u - 2\right)\sqrt{u}\frac{1}{5}du \\
&= \frac{1}{25}\int u^{3/2}du + \frac{2}{5}u^{1/2}du \\
&= \frac{2}{125}u^{5/2} + \frac{4}{15}u^{3/2} + C \\
&= \frac{2}{125}(5x+10)^{5/2} + \frac{4}{15}(5x+10)^{3/2} + C
\end{aligned}$$

Exercise 8. Use u-substitution. Let $u = \ln x$, so $du = \frac{1}{x}dx$. Then

$$\begin{aligned} \int_e^{e^2} \frac{1}{x \ln x} dx &= \int_1^2 \frac{1}{u} du \\ &= \ln u|_1^2 = \ln 2 - \ln 1 = \ln 2 \end{aligned}$$

(Remember that when you do a u-substitution on a definite integral, you need to change the bounds! Since $u = \ln x$, we have $e \mapsto 1$ and $e^2 \mapsto 2$.)

Exercise 9. Use integration by parts. Let $u = x$ and $dv = e^{2x}dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x}$. Putting this together,

$$\begin{aligned} \int xe^{2x}dx &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \end{aligned}$$

Exercise 10. First, use log properties to make the simplification $\int \ln \sqrt{x}dx = \frac{1}{2}\int \ln x dx$. Now integrate by parts. Let $u = \ln x$ and $dv = dx$, so $du = \frac{1}{x}dx$ and $v = x$. Then

$$\begin{aligned} \frac{1}{2}\int \ln x dx &= \frac{1}{2}\left(x \ln x - \int x\frac{1}{x}dx\right) \\ &= \frac{1}{2}(x \ln x - x) + C \end{aligned}$$

Exercise 11. Use integration by parts repeatedly. First, let $u = x^3 + 2x + 4, dv = \sin x dx, du = 3x^2 + 2, v = -\cos x$. Then

$$\int (x^3 + 2x + 4) \sin x \, dx = -(x^3 + 2x + 4) \cos x + \int (3x^2 + 2) \cos x dx$$

Next, $u = 3x^2 + 2, dv = \cos x dx, du = 6x dx, v = \sin x$, so

$$\int (3x^2 + 2) \cos x dx = (3x^2 + 2) \sin x - 6 \int x \sin x dx$$

Last, $u = x, dv = \sin x dx, du = dx, v = -\cos x$, so

$$\int x \sin x dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C$$

Putting it all together gives

$$\int (x^3 + 2x + 4) \sin x \, dx = -(x^3 + 2x + 4) \cos x + (3x^2 + 2) \sin x + 6x \cos x - 6 \sin x$$

Exercise 12. Integrate by parts, taking $u = P(x)$ and $dv = e^x dx$, so $du = P'(x)$ and $v = e^x$. Thus

$$\int P(x)e^x dx = P(x)e^x - \int P'(x)e^x$$

Repeating this process on the new integral, we get

$$\int P(x)e^x dx = P(x)e^x - P'(x)e^x + P''(x)e^x - \cdots + (-1)^n P^{(n)}(x)e^x + (-1)^{n+1}\int P^{(n+1)}(x)e^x dx$$

where $P^{(n)}(x)$ denotes the nth derivative of $P(x)$. By repeating the integration by parts more and more times, one can see this equation holds for any positive integer n. In particular if n is the degree of $P(x)$, then we will have $P^{(n+1)}(x) = 0$, so the last term will be zero, so

$$\int P(x)e^x dx = P(x)e^x - P'(x)e^x + P''(x)e^x - \cdots + (-1)^n P^{(n)}(x)e^x$$

If we repeat essentially the same argument with e^x replaced by $\sin x$, we see that

$$\begin{aligned}\int P(x)\sin x dx &= -P(x)\cos x + \int P'(x)\cos x dx \\ &= -P(x)\cos x + P'(x)\sin x - \int P''(x)\sin x dx \\ &= -P(x)\cos x + P'(x)\sin x + P''(x)\cos x + \int P'''(x)\cos x dx \\ &\vdots\end{aligned}$$

Each four terms will cycle through one of $-\cos x, \sin x, \cos x, -\sin x$, in that order. If n is the degree of the polynomial, then we will have $P^{(n+1)}(x) = 0$, and so the procedure can stop.

If instead of $\sin x$ we started with $\cos x$, the same pattern would emerge, except the terms would go in the order $\sin x, \cos x, -\sin x, -\cos x$.