SOLUTIONS FOR MATH 1B WEEK 6, THURSDAY

Exercise 1. The sequence does converge. Indeed,

$$\lim_{n\to\infty} 3^n 7^{-n} = \lim_{n\to\infty} \left(\frac{3}{7}\right)^n = 0$$

since $|\frac{3}{7}| < 1$. Thus the sequence converges to zero. (I want to make this clear: it converges because the limit *exists*. The fact that it's 0 and not some other number is a coincidence.)

Exercise 2. This sequence converges. First compute

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Then since the function $f(x) = \cos(\pi x)$ is continuous at $x = 1$, we have

$$\lim_{n\to\infty} \cos\left(\frac{n\pi}{n+1}\right) = \cos\left(\pi \lim_{n\to\infty} \frac{n}{n+1}\right) = \cos\pi = -1.$$

Exercise 3. The first step to solving this is to write down a rule for a_n without any "…" Although it may be hard to write one directly, we can define a_n *recursively*, by $a_1 = \sqrt{2}$ and $a_n = \sqrt{2a_{n-1}}$ for $n > 1$.

We can write the first few terms as $2^{1/2}, 2^{3/4}, 2^{7/8}, \ldots$. This leads us to think about the sequence of exponents. To make this precise, let $b_n = \log_2 a_n$. Translating the rules above, we have $b_1 = \frac{1}{2}$ and $b_n = \frac{1}{2}(1 + b_{n-1})$ for $n > 1$. Intuitively, we are repeatedly taking the average of b_n with 1, so this should converge to 1. More precisely, notice that

$$\begin{aligned}
b_1 &= \frac{1}{2} \\
b_2 &= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{4} \\
b_3 &= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\
b_n &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}.
\end{aligned}$$

This is a geometric series, for which we have a nice formula:

$$\lim_{n\to\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}\frac{1}{1-\frac{1}{2}} = 1$$

Now we translate this back to the limit of the a_n's. By how we defined b_n, we have $a_n = 2^{b_n}$. Since $f(x) = 2^x$ is a continuous function,

$$\lim_{n\to\infty} a_n = 2^{\lim_{n\to\infty} b_n} = 2^1 = 2$$

(*Remark*: The argument "averaging with 1 repeatedly should converge to 1" can be made precise by the *contraction mapping principal.* Loosely speaking, it says that if a function f "brings numbers closer together," then there is a unique number x such that $f(x) = x$, and that any sequence of the form $\{a, f(a), f(f(a)), f(f(f(a))), \dots\}$ converges to x. This is an important theorem in analysis which, perhaps unsurprisingly, is proved using a similar geometric series.)

Exercise 4.

(1) Use log properties to get a telescoping series:

$$\sum_{n=1}^{N} \ln \frac{n}{n+1} = \sum_{n=1}^{N} \ln n - \ln(n+1) = \ln 1 - \ln(N+1)$$

As $N \to \infty$, this goes to $-\infty$. Thus the series diverges.

(2) Factor $n^3 - n = n(n+1)(n-1)$. Use a partial fraction decomposition to get a different sort of telescoping series, where all but the first two and last two terms cancel:

$$\sum_{n=2}^{N} \frac{1}{n^3 - n} = \sum_{n=2}^{N} \left(\frac{-1}{n} + \frac{1}{2}\frac{1}{n+1} + \frac{1}{2}\frac{1}{n-1} \right) = \frac{1}{2} - \frac{1}{4} - \frac{1}{2}\frac{1}{N} + \frac{1}{2}\frac{1}{N+1}$$

As $N \to \infty$, this converges to $\frac{1}{4}$.

Exercise 5. Let $f(x) = x^2 e^{-x^3}$. It is clear that f is positive and continuous. To check if it is decreasing, take the first derivative:

$$f'(x) = 2xe^{-x^3} - 3x^4 e^{-x^3} = x(2 - 3x^3)e^{-x^3}$$

This is negative whenever $x \geq \sqrt[3]{2/3}$, so f is eventually decreasing. (Recall that we can change lower bounds without changing convergence, so "eventually decreasing" is always good enough.) Thus we can apply the integral test. We have

$$\int_1^\infty x^2 e^{-x^3} dx = \lim_{a \to \infty} \left(-\frac{1}{3} e^{-a^3} + \frac{1}{3} e^1 \right) = \frac{e}{3}.$$

This means the sum converges.

Exercise 6. Let A be the area of the starting triangle. (We'll compute this at the end, but for now it will just make things messier.) On the first step, the new triangles we add have side lengths scaled down by $1/3$, so they have area $A/3^2 = A/9$. On the n-th step, they have sides scaled by $1/3^n$, so the area of each is $A/9^n$.

Now we have to figure out how many triangles are added on each step, which means we have to know how many sides there are. We start with three sides. After each step, a single side is broken up into 4 sides, so the number of sides multiplies by 4. Thus the number of triangles added on the n-th step is $3 \cdot 4^{n-1}$.

Thus the total area is

$$A + 3\frac{A}{9} + 3 \cdot 4 \cdot \frac{A}{9^2} + 3 \cdot 4^2 \cdot \frac{A}{9^3} + \cdots = A + \frac{3A}{9} \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n = A + \frac{3A}{9} \frac{1}{1 - \frac{4}{9}} = \frac{8}{5}A.$$

Using some high-school geometry, the area of the starting triangle is $\frac{\sqrt{3}}{4}$, so the total area is $\frac{2\sqrt{3}}{5}$.

Exercise 7. From the previous exercise, we know that the number of sides after n steps is $3 \cdot 4^n$. The side lengths start as 1, and scale down by $1/3$ after each step, so the length of the sides after n steps is $1/3^n$. Thus the perimeter after n steps is

$$P(n) = 3 \cdot 4^n \cdot \frac{1}{3^n} = 3 \cdot \left(\frac{4}{3}\right)^n$$

Since $\frac{4}{3} > 1$, this blows up to infinity as $n \to \infty$. We conclude that although the Koch snowflake has finite area, its perimeter is infinite. (You could shade in a picture of one with only a modest amount of paint, but not even with all the paint in the universe could you draw its outline!)