On relations between additive and max -convolutions in classical, free and Boolean probability theory

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Additive convolutions

Free and Boolean independence

In noncommutative probability theory, there are two kinds of independence of noncommutative random variables. They are called free independence and Boolean independence.

Free and Boolean convolutions

Let $\mu, \nu \in \mathcal{P}^1$ and consider freely (resp. Boolean) independent selfadjoint r.v.s. X and Y such that $X \sim \mu$ and $Y \sim \nu$. The distribution of $X + Y$ is called the free (resp. Boolean) convolution of μ and ν, denoted by $\mu \boxplus \nu$ (resp. $\mu \uplus \nu$).

[1] $\mathcal{P}$: the set of all probability measures on $\mathbb{R}$

Voiculescu transform

The Cauchy transform is given by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t), \qquad z \in \mathbb{C}^+.$$

The Voiculescu transform is defined by

$$\varphi_\mu(z) = G_\mu^{-1}\left(\frac{1}{z}\right) - z$$

for z in a suitable domain of $\mathbb{C}^+$.

- For any $\mu, \nu \in \mathcal{P}$, we get $\varphi_{\mu \boxplus \nu} = \varphi_\mu + \varphi_\nu$. (Voiculescu '86, Maassen '92, Bercovici-Voiculescu '93)
- For any $\mu \in \mathcal{P}$ there is $\{\mu_t\}_{t \geq 1}$ s.t. $\varphi_{\mu_t} = t\varphi_\mu$. Denoted by $\mu^{\boxplus t} := \mu_t$ for $t \geq 1$.

Self-energy function

Define

$$F_\mu(z) := \frac{1}{G_\mu(z)}, \quad E_\mu(z) := z - F_\mu(z), \qquad z \in \mathbb{C}^+.$$

For every $\mu \in \mathcal{P}$, we have

$$E_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t} d\rho(t), \qquad z \in \mathbb{C}^+$$

where $\gamma \in \mathbb{R}$ and ρ is a finite measure on $\mathbb{R}$.

- For any $\mu, \nu \in \mathcal{P}$, we get $E_{\mu \boxplus \nu} = E_\mu + E_\nu$. (Speicher, Woroudi, 97)
- For any $\mu \in \mathcal{P}$, there is $\{\mu_t\}_{t \geq 0} \subset \mathcal{P}$ s.t. $E_{\mu_t} = tE_\mu$ and $\mu_0 = \delta_0$. Denoted by $\mu^{\uplus t} := \mu_t$.

Max-convolutions

Classical max-convolution and extreme values

Let X, Y be (classical) random variables with $X \sim \mu$, $Y \sim \nu$.

- $X \vee Y$: the maximum of X, Y.
- F_X: the distribution function of X.
- If X, Y are indep., then $F_{X \vee Y} = F_X F_Y =: F_X \vee F_Y$.
- $\mu \vee \nu((-\infty, \cdot]) := \mu((-\infty, \cdot])\nu((-\infty, \cdot])$.

Extreme values

A nondegenerate distr. funct. F is said to be max-stable if there are a distr. funct. H and $\{a_n\} \subset (0, \infty)$ and $\{b_n\}_n \subset \mathbb{R}$ s.t.

$$H^{\vee n}(a_n \cdot + b_n) \xrightarrow{w} F(\cdot), \qquad n \to \infty.$$

Theorem 1

F is max-stable iff there is $a > 0$ and $b \in \mathbb{R}$ s.t. $F(ax + b)$ is equal to one of the following distributions (extreme values):

$$G(x) := \exp(-\exp(-x)) \quad \textit{(Gumbel)};$$
$$F_\alpha(x) := \exp(-x^{-\alpha}) \text{ for } x > 0 \text{ and } \alpha > 0 \quad \textit{(Fréchet)};$$
$$W_\alpha(x) := \exp(-(-x)^{\alpha}) \text{ for } x \leq 0 \text{ and } \alpha > 0 \quad \textit{(Weibull)}.$$

Spectral maximum

$(\mathcal{A}, \varphi)$: tracial W^*-probability space
X, Y: selfadjoint random variables affiliated with $\mathcal{A}$

- $E_X(B) \in \mathcal{A}$: the spectral projection (B: Borel set in $\mathbb{R}$)
- $X \prec Y$ is defined by $E_X((x,\infty)) \leq E_Y((x,\infty))$ [2]for all $x \in \mathbb{R}$.
- $E_{X \vee Y}((x,\infty)) := E_X((x,\infty)) \vee E_Y((x,\infty))$[3] for $x \in \mathbb{R}$.
- $X \vee Y$ is called the spectral maximum of X, Y.
- The spectral distribution function of X is defined by

$$F_X(x) := \varphi(E_X((-\infty, x])), \qquad x \in \mathbb{R}.$$

[2] P, Q: projections on a Hilbert space $\mathcal{H}$; $P \leq Q \Leftrightarrow P\mathcal{H} \subset Q\mathcal{H}$
[3] $P \vee Q$: the maximum of P, Q w.r.t. $\leq$

Free max-convolution

Let $X \sim \mu, Y \sim \nu$ be freely indep. s.-a. r.v.s affiliated with $\mathcal{A}$.

Theorem 2 (Ben Arous, Voiculescu '06)

$F_{X \vee Y} = \max\{F_X + F_Y - 1, 0\} =: F_X \boxed{\vee} F_Y.$

The free max-convolution is defined by

$$\mu \boxed{\vee} \nu((-\infty, \cdot]) := \mu((-\infty, \cdot]) \boxed{\vee} \nu((-\infty, \cdot]).$$

More generally, we define

$$\mu^{\boxed{\vee} t}((-\infty, \cdot]) := \max\{t\mu((-\infty, \cdot]) - (t-1), 0\}, \qquad t \geq 1.$$

Free extreme values

Theorem 3 (Ben Arous, Voiculescu '06)

A nondegenerate distribution F is free max-stable iff there is $a > 0$ and $b \in \mathbb{R}$ s.t. $F(ax+b)$ is equal to one of the following distributions (free extreme values):

$$E(x) := \max\{0, 1 - e^{-x}\} \quad \textit{(Exponential)};$$
$$P_\alpha(x) := \max\{0, 1 - x^{-\alpha}\} \text{ for } \alpha > 0 \quad \textit{(Pareto)};$$
$$B_\alpha(x) := 1 - |x|^\alpha \text{ for } -1 \le x \le 0 \text{ and } \alpha > 0 \quad \textit{(Beta)}.$$

Boolean max-convolution

$(\mathcal{H}, \xi)$: a Hilbert sp. with unit vector
$X \geq 0$, $Y \geq 0$: Boolean indep. r.v.s. on $(\mathcal{H}, \xi)$
$X \sim \mu \in \mathcal{P}_+$, $Y \sim \nu \in \mathcal{P}_+$[4]

Theorem 4 (Vargas, Voiculescu '17)

$$F_{X \vee Y} = \frac{F_X F_Y}{F_X + F_Y - F_X F_Y} =: F_X \cup\!\!\!\!\vee\, F_Y.$$

The Boolean max-convolution is defined by

$$\mu \cup\!\!\!\!\vee\, \nu([0, \cdot]) := \mu([0, \cdot]) \cup\!\!\!\!\vee\, \nu([0, \cdot]).$$

More generally, we define

$$\mu^{\cup\!\!\!\!\vee\, t}((-\infty, \cdot]) := \frac{\mu([0, \cdot])}{t - (t-1)\mu([0, \cdot])}, \qquad t > 0.$$

[4] $\mathcal{P}_+$: the set of all probability measures on $[0, \infty)$

Boolean extreme values

A nondegenerate distribution F on $[0, \infty)$ is said to be Boolean max-stable if there are a distr. funct. H and $\{a_n\}_n \subset (0, \infty)$ s.t.

$$H^{\cup\!\!\!\vee n}(a_n \cdot) \xrightarrow{w} F(\cdot), \qquad n \to \infty.$$

Theorem 5 (Vargas, Voiculescu '17)

A nondegenerate distribution F on $[0, \infty)$ is Boolean max-stable iff there are $\lambda > 0$ and $\alpha > 0$ s.t. F is the following distribution function.

$$D_{\lambda,\alpha}(x) := (1 + \lambda x^{-\alpha})^{-1} \quad \textit{(Dagum)}.$$

Limit theorems for free multiplicative convolution

Free multiplicative convolution

For $\mu \in \mathcal{P}_+$, $\nu \in \mathcal{P}$, the free multiplicative convolution $\mu \boxtimes \nu \in \mathcal{P}$ is defined as the distribution of $\sqrt{X} Y \sqrt{X}$, where $X \geq 0$ and Y are free random variables with distributions μ and ν, respectively.

S-transform I

Consider $\mu \in \mathcal{P}$. We define

$$\Psi_\mu(z) := \int_0^\infty \frac{xz}{1-xz} d\mu(x).$$

Suppose $\delta_0 \neq \mu \in \mathcal{P}_+$.

- The function Ψ_μ is univalent around $(0, \infty)$ taking values in a neighborhood of $(\mu(\{0\}) - 1, 0)$.
- The S-transform of μ is defined by

$$S_\mu(z) := \frac{z+1}{z} \Psi_\mu^{-1}(z), \qquad z \in (\mu(\{0\}) - 1, 0).$$

S-transform II

For $\mu, \nu \in \mathcal{P}_+ \setminus \{\delta_0\}$, we have

$$S_{\mu \boxtimes \nu}(z) = S_\mu(z) S_\nu(z),$$

for z in the common interval provided μ, ν.

Lemma 1 (Haagerup, Möller, '13)

Consider $\mu \in \mathcal{P}_+$ not being a Dirac measure. Then S_μ is strictly decreasing on $(\mu(\{0\}) - 1, 0)$. Moreover, $S_\mu(\mu(\{0\}) - 1, 0) = (b_\mu^{-1}, a_\mu^{-1})$, where $0 \le a_\mu < b_\mu \le \infty$ are defined by

$$a_\mu := \left(\int_0^\infty x^{-1} d\mu(x) \right)^{-1}, \qquad b_\mu := \int_0^\infty x d\mu(x).$$

Note that if $\mu(\{0\}) > 0$, then we understand $a_\mu^{-1} = \infty$.

S-transform III

Lemma 6 (Belinschi-Nica '08)

For $\delta_0 \neq \mu \in \mathcal{P}_+$, we have

$$S_{\mu^{\boxplus t}}(z) = \frac{1}{t} S_\mu\left(\frac{z}{t}\right), \qquad t \geq 1.$$

$$S_{\mu^{\uplus t}}(z) = \frac{1}{t} S_\mu\left(\frac{z}{t - z + tz}\right), \qquad t > 0.$$

Limit theorem for free multiplicative convolution

Suppose $\mu \in \mathcal{P}_+$. We consider a sequence $(\mu^{\boxtimes n})^{1/n}$. [5]

Proposition 7 (Tucci '10, Haagerup-Möller '13)

The sequence weakly converges as $n \to \infty$ Write $\Phi(\mu)$ the weak limit of the sequence.

- *If μ is a Dirac measure on $[0, \infty)$, then $\Phi(\mu) = \mu$.*
- *If μ is not a Dirac measure on $[0, \infty)$, then it satisfies*

$$\Phi(\mu)(\{0\}) = \mu(\{0\}), \quad \Phi(\mu)\left(\left[0, \frac{1}{S_\mu(x-1)}\right]\right) = x,$$

for any $x \in (\mu(\{0\}), 1)$. The support of $\Phi(\mu)$ is the closure of (a_μ, b_μ).

[5] μ^k: the distribution of X^k, where $X \sim \mu$.

Lemma 8

For all $\mu \in \mathcal{P}_+$ *not being a Dirac measure and for all* $x \in (a_\mu, b_\mu)$, *we have*

$$\Phi(\mu)([0,x]) = S_\mu^{-1}\left(\frac{1}{x}\right) + 1,$$

and therefore $(a_\mu, b_\mu) = \{x : \Phi(\mu)([0,x]) \in (\mu(\{0\}), 1)\}$.

Calculation

- Let π be the (standard) Marchenko-Pastur distribution. Then

$$\Phi(\pi)([0,x]) = x, \qquad x \in (0,1).$$

- Let f_α and b_α be strictly positive free stable law and strictly positive Boolean stable law with index $\alpha \in (0,1)$, respectively. Then

$$\Phi(f_\alpha)([0,\cdot]) = P_{\frac{\alpha}{1-\alpha}}(\cdot) \qquad \text{(Pareto, Free extreme).}$$

$$\Phi(b_\alpha)([0,\cdot]) = D_{1,\frac{\alpha}{1-\alpha}}(\cdot) \qquad \text{(Dagum, Boolean extreme).}$$

Main theorem

Main theorem

Theorem 9 (U.)

Consider $\mu \in \mathcal{P}_+$. Then we get

- $\Phi(D_{1/t}(\mu^{\boxplus t})) = \Phi(\mu)^{\boxed{\vee} t}$, $t \geq 1$.
- $\Phi(D_{1/t}(\mu^{\uplus t})) = \Phi(\mu)^{\cup\!\!\!\vee t}$, $t > 0$.
- $B_t^{\vee} \circ \Phi = \Phi \circ B_t$, $t \geq 0$, *where*

$$B_t(\mu) := (\mu^{\boxplus(1+t)})^{\uplus \frac{1}{1+t}}, \quad \mu \in \mathcal{P} \quad \textit{(Belinschi-Nica '08)}$$

$$B_t^{\vee}(\mu) := (\mu^{\boxed{\vee}(1+t)})^{\cup\!\!\!\vee \frac{1}{1+t}}, \quad \mu \in \mathcal{P}^{+} \quad \textit{(U. '19)}$$

Outline of Proof (1st statement)

Fix $t \geq 1$. If μ is a Dirac measure, then $\Phi(D_{1/t}(\mu^{\boxplus t})) = \Phi(\mu)^{\boxed{\vee} t}$. Therefore we may assume that μ is not a Dirac measure. Moreover, we assume that $\mu(\{0\}) = 0$.

- $\mu^{\boxplus t}(\{0\}) = \mu^{\boxed{\vee} t}(\{0\}) = 0$.
- By Lemma 8, the support of $\Phi(\mu)^{\boxed{\vee} t}$ is the closure of

$$M_t := \{x : \Phi(\mu)^{\boxed{\vee} t}([0, x]) \in (0, 1)\}.$$

- We have

$$\begin{aligned} A_t : &= \{x : \Phi(D_{1/t}(\mu^{\boxplus t}))([0, x]) \in (0, 1)\} \\ &= \left(\frac{a_{\mu^{\boxplus t}}}{t}, \frac{b_{\mu^{\boxplus t}}}{t}\right). \end{aligned}$$

We would like to show that

- $A_t = M_t$
- For all $x \in A_t = M_t$, we get

$$\Phi(D_{1/t}(\mu^{\boxplus t}))([0,x]) = t\Phi(\mu)([0,x]) - (t-1).$$

Proof of $M_t \subset A_t$ and 2nd claim

For all $x \in M_t$, we get $t\Phi(\mu)([0,x]) - (t-1) \in (0,1)$. Since

$$M_t = \left\{ x : \Phi(\mu)([0,x]) \in \left(1 - \frac{1}{t}, 1\right) \right\} \subset (a_\mu, b_\mu),$$

we have

$$x = \frac{1}{S_\mu(\Phi(\mu)([0,x])) - 1}$$

by Lemma 8.

Therefore we get [6]

$$\begin{aligned}\Phi(\mu^{\boxplus t})([0,tx]) &= \Phi(\mu^{\boxplus t})\left(\left[0, \frac{t}{S_\mu(\Phi(\mu)([0,x])-1)}\right]\right)\\ &= \Phi(\mu^{\boxplus t})\left(\left[0, \frac{1}{S_{\mu^{\boxplus t}}\left(t(\Phi(\mu)([0,x])-1)\right)}\right]\right)\\ &= \Phi(\mu^{\boxplus t})\left(\left[0, \frac{1}{S_{\mu^{\boxplus t}}\left(\{t\Phi(\mu)([0,x])-(t-1)\}-1\right)}\right]\right)\\ &= t\Phi(\mu)([0,x])-(t-1).\end{aligned}$$

Therefore $\Phi(D_{1/t}(\mu^{\boxplus t})([0,x]) \in (0,1)$, so $x \in A_t$.

[6] $S_{\mu^{\boxplus t}}(z) = \frac{1}{t} S_\mu\left(z/t\right)$

Proof of $A_t \subset M_t$

We show that $x \in (a_{\mu^{\boxplus t}}, b_{\mu^{\boxplus t}})$ [7] implies that $x/t \in M_t$. For any $x \in (a_{\mu^{\boxplus t}}, b_{\mu^{\boxplus t}})$, we have

$$\Phi(\mu^{\boxplus t})([0,x]) = S^{-1}_{\mu^{\boxplus t}}\left(\frac{1}{x}\right) + 1 = tS^{-1}_{\mu}\left(\frac{t}{x}\right) + 1.$$

Therefore

$$x = \frac{t}{S_\mu\left(\frac{1}{t}\Phi(\mu^{\boxplus t})([0,x]) - \frac{1}{t} + 1 - 1\right)}.$$

[7] $(a_\mu, b_\mu) = \{x : \Phi(\mu)([0,x]) \in (\mu(\{0\}), 1)\}$

Hence

$$\Phi(\mu)([0,x/t]) = \Phi(\mu)\left(\left[0, \frac{1}{S_\mu\left(\frac{1}{t}\Phi(\mu^{\boxplus t})([0,x]) - \frac{1}{t} + 1 - 1\right)}\right]\right)$$
$$= \frac{1}{t}\Phi(\mu^{\boxplus t})([0,x]) - \frac{1}{t} + 1 \in \left(1 - \frac{1}{t}, 1\right).$$

Hence $x/t \in M_t$ [8]. Thus

$$A_t = \left(\frac{a_{\mu^{\boxplus t}}}{t}, \frac{b_{\mu^{\boxplus t}}}{t}\right) \subset M_t.$$

Finally, we have $A_t = M_t$.

[8] $M_t = \{x : \Phi(\mu)([0,x]) \in (1 - 1/t, 1)\}$

Proof (3rd statement)

If μ is a Dirac measure, then $\Phi(B_t(\mu)) = \mu = B_t^{\vee}(\Phi(\mu))$. Therefore we may assume that μ is not a Dirac measure. By main theorem, we have

$$\begin{aligned}
\Phi(B_t(\mu)) &= D_{\frac{1}{1+t}}\left(\Phi(\mu^{\boxplus(1+t)})^{\Cup \frac{1}{1+t}}\right) \\
&= D_{\frac{1}{1+t}}\left(\left(D_{1+t}\left(\Phi(\mu)^{\boxtimes(1+t)}\right)\right)^{\Cup \frac{1}{1+t}}\right) \\
&= D_{\frac{1}{1+t}} \circ D_{1+t}\left(\left(\Phi(\mu)^{\boxtimes(1+t)}\right)^{\Cup \frac{1}{1+t}}\right) = B_t^{\vee}(\Phi(\mu)).
\end{aligned}$$

Ψ-operator

Define the following operator:[9]

$$\Psi := \mathcal{X}^{\vee} \circ \Phi \circ \mathcal{X}^{-1} = \mathcal{X}^{\vee} \circ \Phi \circ B_1^{-1} \circ \Lambda.$$

Theorem 10 (U.)

For any infinitely divisible distribution $\mu \in \mathcal{P}_+$, we have

$$\Psi(D_{1/t}(\mu^{*t})) = \Psi(\mu)^{\vee t}, \qquad t > 0.$$

[9] Λ: Bercovici-Pata bijection
$\mathcal{X}$: Boolean-classical Bercovici-Pata bijection
$\mathcal{X}^{\vee}(\mu)([0,x]) = \exp(1 - \mu([0,x])^{-1})$

Examples

- Let c_α be strictly positive classical stable law with index $\alpha \in (0,1)$. Then

$$\Psi(c_\alpha)([0,\cdot]) = F_{\frac{\alpha}{1-\alpha}}(\cdot) \qquad \text{(Fréchet, Classical extreme)}$$

- Denote by $Po(\lambda)$ the Poisson law with parameter $\lambda > 0$, that is,

$$Po(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \delta_k.$$

Then

$$\Psi(Po(\lambda))([0,x]) = \exp(-\lambda + x), \qquad x \in (0,\lambda).$$

Problems

- For different $\mu, \nu \in \mathcal{P}_+$, can we get formula of $\Phi(\mu \boxplus \nu)$, $\Phi(\mu \uplus \nu)$ and $\Psi(\mu * \nu)$?
- Can we understand tail behavior of $\Phi(\mu)$ and $\Psi(\mu)$ if μ is regularly varying with index α?
- Let γ_p be the gamma distribution. What is $\Psi(\gamma_p)$?
- Let $\beta_{p,q}$ be the beta law. What is $\Psi(\beta_{p,q})$?
- What does Ψ mean in terms of classical probability theory?