Positive definite reflection length in type B

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Let G be a group. A function $\phi\colon G \to \mathbb{C}$ is *positive definite* if for any number $k \in \mathbb{N}$

$$\sum_{i,j=1}^{k} z_i \overline{z_j} \phi(g_j^{-1} g_i) \geq 0$$

for all $z_1, \ldots, z_k \in \mathbb{C},\ g_1, \ldots, g_k \in G$.

1. 1979 – Haagerup proved that the function

$$g \to q^{\ell_S(g)}$$

is positive definite for $-1 \leq q \leq 1$ on the free group F_N, for $N \geq 2$, where

$$\ell_S := \text{is the minimal number of generators}$$

Case $N = 1$ was done by Poisson.

2. 1988 – Bożejko, Januszkiewicz and Spatzier, were studying similar problem and they proved that the function $g \to q^{\ell_S(g)}$ is positive definite for all Coxeter groups.

3. 1996, 2003 – This result was generalized to multi-parameters and also other variants of the Coxeter function (colour-length) by Bożejko, Szwarc and Speicher

All the considered functions share two properties

1. they are positive definite on the continuos set $-1 \leq q \leq 1$,
2. they are not (generically) invariant by conjugation i.e.

$$\text{it is not true that } \phi(g) = \phi(hgh^{-1}) \text{ for any } g, h \in G.$$

The most natural way to modify the Coxeter function in order to obtain its analog which is central on G is to replace the Coxeter length ℓ_S by the *reflection length* $\ell_{\mathcal{R}}$,

$$\ell_{\mathcal{R}} = \text{the minimal number of reflections}$$

Central functions

1. 1964 Thoma obtain complete characterization of central normalized positive defined function in the case of the infinite symmetric group $\mathfrak{S}_\infty$
2. 1974, 1976 Voiculescu in the case of infinite dimensional Lie groups $U(\infty), SO(\infty)$
3. 1981 Vershik and Kerov

Motivation

In the case of $G = \mathfrak{S}_\infty$ this length $\ell_{\mathcal{R}}(\sigma)$ is given by the minimal number of transpositions

$$\begin{aligned}\ell_{\mathcal{R}}(\sigma) : &= \min\{\tau_1, \ldots, \tau_n \in \mathcal{T} : \sigma = \tau_1 \cdots \tau_n\} \\ &= n - \text{ number of cycles of } \sigma.\end{aligned}$$

where $\mathcal{T}$ is the set of all transpositions.
From Thoma result follows that

$$f_q(\sigma) := q^{\ell_{\mathcal{R}}(g)}$$

is positive definite if and only if

$$q = \frac{\epsilon}{N},\ N \in \mathbb{N} \text{ and } \epsilon \in \{-1, 0, 1\}.$$

Bożejko and Guta in 2001 used this positive definite function to construct a Gaussian operator.

Coxeter group of type B

The Coxeter group of type B $B(n)$ (= hyperoctahedral group) is the group of permutations on

$$\{\bar{n}, \dots, \bar{1}, 1, \dots, n\}$$

satisfying $\sigma(\bar{i}) = \bar{\sigma}(i)$, where we use the convention that $\bar{i} = -i$ for example

$$\bar{1} = -1$$
$$\overline{-1} = 1.$$

Equivalently $B(n)$ is the group of symmetries of the n-dimensional hypercube

$$B(n) = \{\sigma \in S(\pm 1, \dots, \pm n) \mid \sigma(-i) = -\sigma(i)\}.$$

Example $B(6)$

$$\sigma = \begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{6} & \bar{3} & \bar{1} & 5 & 4 & 2 & \bar{2} & \bar{4} & \bar{5} & 1 & 3 & 6 \end{pmatrix}$$

We have two types of cycles:

1. cycles which do not contain i and $\bar{i}$ for any i,
2. cycles in which i is an element if and only if $\bar{i}$ is an element.

Cycles of the first type come in natural pairs, and instead of

$$(i_1, i_2, \ldots, i_k)(\bar{i}_1, \bar{i}_2, \ldots, \bar{i}_k),$$

we write $(i_1, i_2, \ldots, i_k)$ and call it a positive cycle.

Cycles of the second type are of the form

$$(i_1, i_2, \ldots, i_k, \bar{i}_1, \bar{i}_2, \ldots, \bar{i}_k).$$

We shorten that to $(i_1, i_2, \ldots, i_k)^-$ and call it a negative cycle.

For example, the permutation

$$\bar{4} \mapsto \bar{2},\ \bar{3} \mapsto 1,\ \bar{2} \mapsto 4,\ \bar{1} \mapsto 3,\ 1 \mapsto \bar{3},\ 2 \mapsto \bar{4},\ 3 \mapsto \bar{1},\ 4 \mapsto 2$$

is written as $(1,\bar{3})(\bar{1},3)(2,\bar{4},\bar{2},4) = (1,\bar{3})(2,\bar{4})^{-}$.

Example $B(6)$

$$\sigma = \begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{6} & \bar{3} & \bar{1} & 5 & 4 & 2 & \bar{2} & \bar{4} & \bar{5} & 1 & 3 & 6 \end{pmatrix}$$

$$\sigma = (1, \bar{2}, 4)(\bar{1}, 2, \bar{4})(3, \bar{5}, \bar{3}, 5)(6)(\bar{6}) = (1, \bar{2}, 4)(3, \bar{5})^{-}(6)$$

The conjugacy classes of $B(n)$ are identified with pairs of partitions

$$(\rho^+, \rho^-) = (\rho_1^+ \dots \rho_k^+, \rho_1^- \dots \rho_m^-)$$

of total size at most n, where the first partition ρ^+ has no parts equal to 1, i.e.

$$|\rho^+| + |\rho^-| = \sum_i \rho_i^+ + \sum_j \rho_j^- \leq n; \qquad \rho_i^+ > 1$$

For example, the conjugacy class of

- $(1,5,\bar{2})(4,7)(6;\bar{8})^{-}(3)^{-}$ is $(32;21) \subset B(6)$;
- $(1,5,\bar{2})(9,10,11)(4,7)(6;\bar{8})^{-}(3)^{-}$ is $(332;21) \subset B(11)$;
- $(1,5,\bar{2})(\bar{9},11)(4,7)(10)(6;\bar{8})^{-}(3)^{-}$ is $(322;21) \subset B(11)$.

Reflections=Transpositions

Positive reflections $(i,j)(\bar{i},\bar{j})$ for $i \neq \bar{j}$ we denote it by $\mathcal{R}_+$

Negative reflections $(i,\bar{i})$ we denote it by $\mathcal{R}_-$

Suppose that $\sigma \in B(n)$ is expressed as a product of reflections, where the number of reflections is minimal in non-mixed factorization

$$\sigma = r_1 \cdots r_k, \qquad r_i \in \mathcal{R},$$

Def. non-mixed factorization means that

$$r_i \cap r_j = \emptyset \text{ for all reflections } r_i \text{ and } r_j \text{ appearing in } \sigma.$$

Let

$$\ell_{\mathcal{R}_+}(\sigma) = \text{The number of positive reflections } r_i \text{ appearing in the minimal, non-mixed factorization,}$$

$$\ell_{\mathcal{R}_-}(\sigma) = \text{The number of negative reflections } r_i \text{ appearing in the minimal, non-mixed factorization.}$$

We define the *signed reflection function* by

$$\phi_{q_+,q_-} : B(n) \to \mathbb{C}$$
$$\phi_{q_+,q_-}(\sigma) := q_+^{\ell_{\mathcal{R}_+}(\sigma)} q_-^{\ell_{\mathcal{R}_-}(\sigma)},$$

were $q_+, q_- \in \mathbb{C}$ be parameters.

Remark

We can not put

$$\ell_{\mathcal{R}_+}(\sigma) = \text{The minimal number of positive reflections } r_i \text{ appearing in the factorization of } \sigma,$$

$$\ell_{\mathcal{R}_-}(\sigma) = \text{The minimal number of negative reflections } r_i \text{ appearing in the factorization of } \sigma.$$

which is direct analog of $\ell_{\mathcal{R}}(g)$.

To see this, we consider

$$\sigma = \begin{pmatrix} \bar{2} & \bar{1} & 1 & 2 \\ 2 & 1 & \bar{1} & \bar{2} \end{pmatrix} \in B(2)$$

which is the product of two negative reflections

$$\sigma = (1, \overline{1})(2, \overline{2})$$

but also as the product of two positive reflections

$$\sigma = (1, 2)(\overline{1}, \overline{2})(1, \overline{2})(\overline{1}, 2).$$

Note that we have a ascending tower of groups:

$$B(1) < B(2) < \dots,$$

which allows to define the infinite group $B(\infty)$ as the inductive limit of this tower.

Definition

A *character* $\phi : B(\infty) \to \mathbb{C}$ is a central, positive-definite function which takes value 1 on the identity.

Definition

A character $\phi : B(\infty) \to \mathbb{C}$ is called *extreme* if it is extreme point of all normalized positive-definite central function on the group.

Theorem

Let $q_+, q_- \in \mathbb{C}$. The following conditions are equivalent:

1. *The function ϕ_{q_+,q_-} is positive definite on $B(\infty)$;*
2. *The function ϕ_{q_+,q_-} is a character of $B(\infty)$;*
3. *The function ϕ_{q_+,q_-} is an extreme character of $B(\infty)$;*
4. *for $M, N \in \mathbb{N}, M + N \neq 0, \epsilon \in \{1, -1\}$*

$$q_+ = \frac{\epsilon}{M+N}, q_- = \frac{M-N}{M+N} \qquad \text{discrete,}$$

$$\text{or } q_+ = 0, -1 \leq q_- \leq 1 \qquad \text{continuous.}$$

Proof:

- uses a representation theory of $B(n)$;
- Frobenius formula;

We can apply the Frobenius formula to show that the reflection function $g \to q^{\ell_{\mathcal{R}}(g)}$ on the infinite symmetric group $\mathfrak{S}_\infty$ is positive definite if and only if $q = \frac{\epsilon}{N}$, $N \in \mathbb{N}$ and $\epsilon \in \{-1, 0, 1\}$.

We assume that the parameters q_+ and q_- are as in the main Theorem.

Let $H_{\mathbb{R}}$ be a separable real Hilbert space and let H be its complexification with the inner product $\langle\cdot,\cdot\rangle$.

We consider the Hilbert space $\mathcal{K} := H \otimes H$, with the inner product

$$\langle x \otimes y, \xi \otimes \eta\rangle_{\mathcal{K}} = \langle x, \xi\rangle\langle y, \eta\rangle.$$

We define a natural action of $B(n)$ on $\mathcal{K}_n := H^{\otimes 2n}$ by setting:

$$\sigma : \mathcal{K}_n \to \mathcal{K}_n$$

$$x_{\overline{n}} \otimes \cdots \otimes x_{\overline{1}} \otimes x_1 \otimes \cdots \otimes x_n \mapsto x_{\sigma(\overline{n})} \otimes \cdots \otimes x_{\sigma(\overline{1})} \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

1. $F := \bigoplus_{n=0}^{\infty} \mathcal{K}_n = \bigoplus_{n=0}^{\infty} H^{\otimes 2n}$;
2. $P_{q_+,q_-}^{(n)} := \sum_{\sigma \in B(n)} \phi_{q_+,q_-}(\sigma)\,\sigma, \qquad n \geq 1$;
3. For $\mathbf{x} \in \mathcal{K}_n$ and $\mathbf{y} \in \mathcal{K}_m$ we deform inner product by
$$\langle \mathbf{x}, \mathbf{y} \rangle_{q_+,q_-} := \delta_{n,m} \langle \mathbf{x}, P_{q_+,q_-}^{(m)} \mathbf{y} \rangle_{0,0}$$
4. $\mathcal{F}_{q_+,q_-}(\mathcal{K})$ is denote the algebraic full Fock space with the inner product $\langle \cdot, \cdot \rangle_{q_+,q_-}$
5. For $x \otimes y \in \mathcal{K}$ we define
$$b_{q_+,q_-}^{*}(x \otimes y)\mathcal{K}_n \to \mathcal{K}_{n+1}$$
$$\eta \mapsto x \otimes \eta \otimes y.$$
and $b_{q_+,q_-}(x \otimes y)$ be its adjoint operator with respect to the inner product $\langle \cdot, \cdot \rangle_{q_+,q_-}$.

The cyclic commutation relation of type B

For $x \otimes y, \xi \otimes \eta \in \mathcal{K}$ we have

$$b_{q_+,q_-}(x \otimes y) b^*_{q_+,q_-}(\xi \otimes \eta) = \langle x, \xi \rangle \langle y, \eta \rangle \,\mathrm{id} + q_- \langle x, \eta \rangle \langle y, \xi \rangle \,\mathrm{id} + \Gamma_{q_+}(|\xi\rangle\langle x| \otimes |\eta\rangle\langle y|).$$

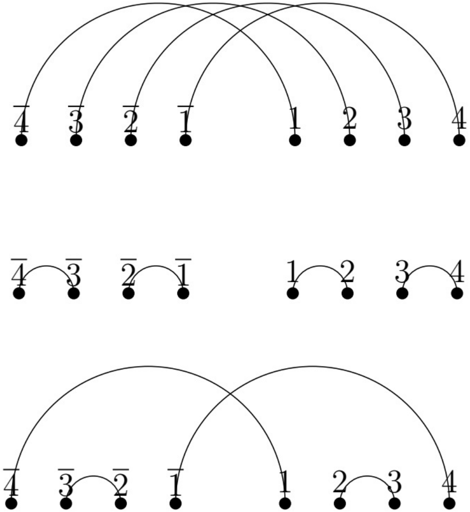
where Γ_{q_+} is the deformation of differential second quantisation operator.

We denote by $\mathcal{P}_2^{sym}(n)$ the subset of pair partitions of

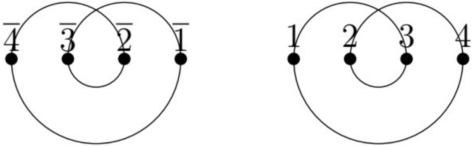
$$\bar{n}, \ldots, \bar{1}, 1, \ldots, n,$$

whose every block is pair such that they are symmetric $\overline{\pi} = \pi$, but every pair $B \in \pi$ is different then its symmetrization $\overline{B}$, i.e.. $B \neq \overline{B}$.

Cycles on pair partitions of type B

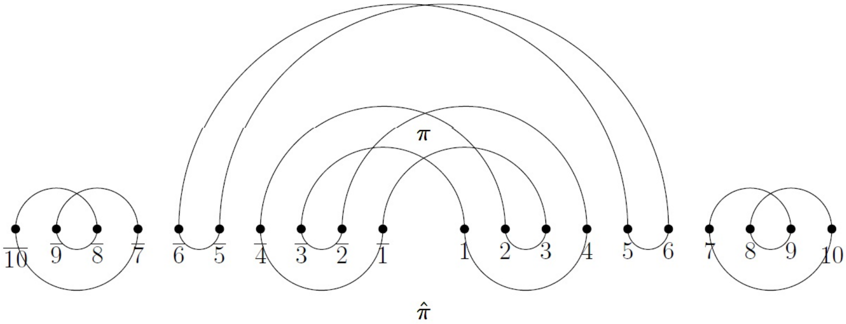


Let $\pi \in \mathcal{P}_2^{sym}(n)$. There exists a unique non-crossing partition $\hat{\pi} \in \mathcal{P}_2^{sym}(n)$, such that the positive/negative pairs of π and $\hat{\pi}$ are coincide;



1. the set of right legs of the positive pairs of π and $\hat{\pi}$ coincide;
2. the set of left legs of the negative pairs of π and $\hat{\pi}$ coincide;

We distinguish two different kinds of cycles: positive and negative, which resembles the description of the cycles in the $B(n)$



$$\mathrm{Cyc}(\pi) = \{(\bar{1}, 3, 2, \bar{4})^{+}, (7, 9, 8, 10)^{+}, (\bar{5}, 6)^{-}\}$$

The operator

$$G(x \otimes y) = b_{q_+,q_-}(x \otimes y) + b^*_{q_+,q_-}(x \otimes y), \quad x, y \in H_{\mathbb{R}},$$

is called the *cyclic Gaussian operator of type B*.

Wick formula

Suppose that $x_1, \ldots, x_{2n} \in H_{\mathbb{R}}$, $x_{\bar{1}}, \ldots, x_{\overline{2n}} \in H_{\mathbb{R}}$, then

$$\varphi(G(x_{\overline{2n}}\otimes_{2n}) \ldots G(x_{\bar{1}} \otimes x_1)) = \sum_{\pi \in \mathcal{P}_2^{sym}(2n)} q_-^{negc(\pi)} q_+^{n-c(\pi)} \times \prod_{\{i,j\} \in \text{Pair}(\pi)} \langle x_i, x_j \rangle,$$

where

1. $c(\pi)$ is the number of cycles of π;
2. $negc(\pi)$ is the number of negative cycle of π;

The Askey-Wimp-Kerov distribution ν_c is the measure on $\mathbb{R}$, with Lebesgue density

$$\frac{1}{\sqrt{2\pi}\Gamma(c+1)}\left|D_{-c}(ix)\right|^{-2} \qquad x \in \mathbb{R}, \quad c \in (-1,\infty)$$

where $D_{-c}(z)$ is the solution to the differential Weber equation:

$$\frac{d^2y}{dz^2} + \left(\frac{1}{2} - c - \frac{z^2}{4}\right) y = 0,$$

satisfying the initial conditions:

$$D_{-c}(0) = \frac{\Gamma\left(\frac{1}{2}\right) 2^{-c/2}}{\Gamma\left(\frac{1+c}{2}\right)} \text{ and } D'_{-c}(0) = \frac{\Gamma\left(-\frac{1}{2}\right) 2^{-(c+1)/2}}{\Gamma\left(\frac{c}{2}\right)}.$$

The orthogonal polynomials $(H_n(t))_{n=0}^{\infty}$, with respect to ν_c are given by the recurrence relation:

$$tH_n(t) = H_{n+1}(t) + (n+c)H_{n-1}(t), \qquad n = 0, 1, 2, \ldots$$

with $H_{-1}(t) = 0$, $H_0(t) = 1$.

Let μ_{q_+,q_-} be the probability distribution of $G(x \otimes x)$, with respect to the vacuum state. Then μ_{q_+,q_-} is equal to:

- Askey-Wimp-Kerov distribution for $q_+ > 0$;
- the semi-circle distribution for $q_+ = 0$;
- discrete measure of finite support for $q_+ < 0$;

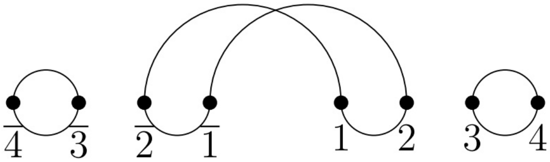
From above, we conclude

$$\#\{\mathrm{Cyc}(\pi) \mid \pi \in \mathcal{P}_2^{sym}(2n)\} = \frac{(2n)!}{n!} = 2n \text{ moment of } N(0,2).$$

Another interesting specialization is given by $q_+ = 0$, which gives us

$$\sum_{\substack{\pi \in \mathcal{P}_2^{sym}(2n): \\ \pi \text{ contains cycles of size 2}}} q_-^{negc(\pi)} = C_n(1+q_-)^n$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$.



Thank you for your attention