Infinite quantum permutations

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Classical permutations

Classical permutations

Permutation matrices can be characterised in the following way.

Lemma

A matrix $u = (u_{ij}) \in M_n(\mathbb{C})$ corresponds to a permutation if and only if the following conditions are satisfied.

- *We have $u_{ij}^2 = u_{ij}$ for all $1 \leq i, j \leq n$.*
- *We have*
$$\sum_{k=1}^{n} u_{ik} = 1 = \sum_{k=1}^{n} u_{kj}$$
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Key idea

The idea behind the theory of quantum permutations is to study more general "permutation matrices", by considering the above relations not over $\mathbb{C}$, but for matrices with entries in more general (noncommutative) algebras.

Quantum permutations

Quantum permutations

Definition (Wang 1998)

A *quantum permutation* of $\mathbf{n} = \{1, \ldots, n\}$ is a pair $(\mathcal{H}, u)$ of a Hilbert space $\mathcal{H}$ and an $n \times n$-matrix matrix $u = (u_{ij})$ of elements $u_{ij} \in B(\mathcal{H})$ such that

- We have $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $1 \leq i, j \leq n$.
- We have
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Matrices $u = (u_{ij})$ satisfying these conditions are also called *magic unitaries*.

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Lemma

Quantum permutations of $\mathbf{n}$ *whose underlying Hilbert space is* $\mathbb{C}$ *are the same thing as permutations of* $\mathbf{n}$.

If $(\mathcal{H}, u)$ is a quantum permutation then the entries in each row/column of u commute. We call $(\mathcal{H}, u)$ *classical* if *all* matrix entries mutually commute.

Examples

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Lemma

Every quantum permutation of $\mathbf{n}$ *for* $n = 1, 2, 3$ *is classical.*

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Proof.

This is obvious for $n = 1$. For $n = 2$ notice that we must have

$$u = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

for some projection p.
For $n = 3$ it is enough to show that u_{ij} and u_{kl} commute provided $i \neq j$ and $k \neq l$. Consider e.g. u_{11} and u_{22}. We get

$$u_{11}u_{22}u_{13} = u_{11}(1 - u_{21} - u_{23})u_{13} = 0,$$

which implies $u_{11}u_{22} = u_{11}u_{22}(u_{11} + u_{12} + u_{13}) = u_{11}u_{22}u_{11}$. This yields

$$u_{11}u_{22} = u_{11}u_{22}u_{11} = (u_{11}u_{22}u_{11})^* = (u_{11}u_{22})^* = u_{22}u_{11}$$

as required. □

Examples

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As soon as $n \geq 4$ one can find non-classical quantum permutations.

Indeed, for arbitrary projections p, q the matrix

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

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Theorem (Banica-Bichon 2009)

For $n = 4$ the irreducible quantum permutations of $\mathbf{n}$ have dimension $1, 2$ or 4, and there are uncountably many (non-isomorphic) such quantum permutations.

Here a quantum permutation $(\mathcal{H}, u)$ is called *irreducible* if the only operators in $B(\mathcal{H})$ commuting with all u_{ij} are scalar multiples of the identity.

Quantum symmetries of graphs

By a (finite) *graph* $X = (I_X, E_X)$ we mean an undirected simple (finite) graph without self-edges. Here I_X is the vertex set and $E_X \subset I_X \times I_X$ the edge set.

The *adjacency matrix* of X is the matrix $A_X \in M_{I_X}(\{0,1\})$ determined by

$$(A_X)_{x,y} = 1 \Leftrightarrow (x,y) \in E_X$$

for all $x, y \in I_X$.

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Terminology

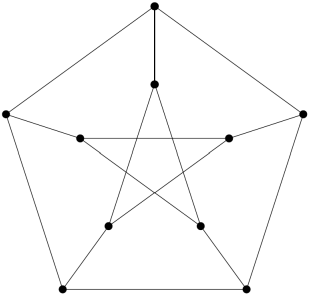
We say that X *has quantum symmetry* if X admits a non-classical quantum automorphism, and that X *has no quantum symmetry* otherwise.

Graphs with no quantum symmetry

Graphs with no quantum symmetry

Theorem (Schmidt 2018)

The Petersen graph has no quantum symmetry.



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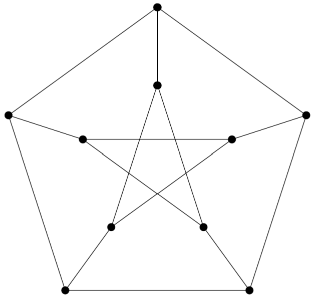

Theorem (Lupini-Mančinska-Roberson 2017)

Almost all finite graphs have no quantum symmetry.

Examples of graphs with quantum symmetry

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Two automorphisms σ, τ of a graph X are called *disjoint* iff $\sigma(i) \neq i$ implies $\tau(i) = i$ and $\tau(i) \neq i$ implies $\sigma(i) = i$.

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Theorem (Junk-Schmidt-Weber 2019)

Almost all finite trees have quantum symmetry.

An intriguing example

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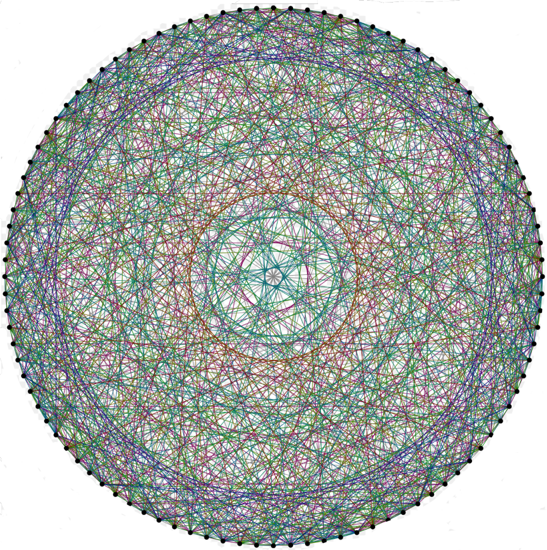


An intriguing example

The *Higman-Sims* graph HS is a graph with the following properties:

- HS has 100 vertices
- Every vertex has 22 neighbours
- HS is triangle-free
- If $a \neq b$ and $a \sim b$ then there are exactly 6 vertices c such that $a \sim c, b \sim c$
- If $a, b, c,$ are distinct and not connected then they have two common neighbours.

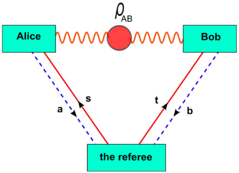
Theorem (Jaeger-Kuperberg - and Jones-Snyder-Edge-Vaes...)

The quantum automorphism group $\mathrm{Qut}(HS)$ *is monoidally equivalent to* $SO_q(5)$ *where* $q = (\frac{1+\sqrt{5}}{2})^2$.

The graph isomorphism game

The graph isomorphism game

This is game is played by two cooperating players, Alice and Bob, against a referee.



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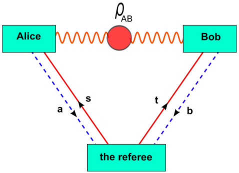

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A *classical* winning strategy exists if and only if X and Y are isomorphic.

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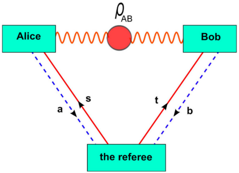

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A *classical* winning strategy exists if and only if X and Y are isomorphic.

Theorem (Brannan-Chirvasitu-Eifler-Harris-Paulsen-Su-Wasilewski 2018)

Two finite graphs X, Y are quantum isomorphic if and only if there exists a perfect quantum strategy for winning the graph isomorphism game.

Here X, Y are called *quantum isomorphic* iff $I_X = I_Y$ and there exists a quantum permutation $(\mathcal{H}, u)$ such that $uA_X = A_Y u$.

The graph isomorphism game

Key point

Quantum isomorphism is strictly weaker than isomorphism!

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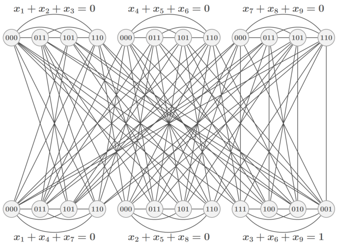

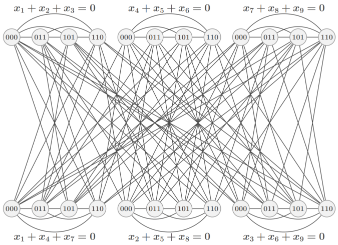

Infinite quantum permutations

Let us extend our considerations to *arbitrary* sets.

Definition

Let I be a set. A *quantum permutation* of I is a pair $(\mathcal{H}, u)$ consisting of a Hilbert space $\mathcal{H}$ and a family $u = (u_{ij})_{i,j \in I}$ of elements $u_{ij} \in B(\mathcal{H})$ such that

- We have $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i, j \in I$.
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Lemma

Quantum permutations of a set I whose underlying Hilbert space is $\mathbb{C}$ are the same thing as permutations of I.

Some further definitions

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Fix a set I throughout.

- If $\sigma = (\mathcal{H}, u)$ and $\tau = (\mathcal{K}, v)$ are quantum permutations then an *intertwiner* from σ to τ is a bounded linear operator $T : \mathcal{H} \to \mathcal{K}$ such that $Tu_{ij} = v_{ij}T$ for all $i, j \in I$.
- The *direct sum* of quantum permutations $\sigma = (\mathcal{H}, u)$ and $\tau = (\mathcal{K}, v)$ is defined by $\sigma \oplus \tau = (\mathcal{H} \oplus \mathcal{K}, u \oplus v)$, where $(u \oplus v)_{ij} = u_{ij} \oplus v_{ij}$ for all $i, j \in I$.
- The *tensor product* of quantum permutations $\sigma = (\mathcal{H}_\sigma, u^\sigma)$ and $\tau = (\mathcal{H}_\tau, u^\tau)$ is defined by $\sigma \otimes \tau = (\mathcal{H}_\sigma \otimes \mathcal{H}_\tau, u^\sigma \mathbin{\bigcirc\!\!\!\!\top} u^\tau)$ where $(u^\sigma \mathbin{\bigcirc\!\!\!\!\top} u^\tau)_{ij} = \sum_{k \in I} u^\sigma_{ik} \otimes u^\tau_{kj}$ for all $i, j \in I$.
- The *contragredient* $\overline{\sigma} = (\mathcal{H}_{\overline{\sigma}}, u^{\overline{\sigma}})$ of a quantum permutation $\sigma = (\mathcal{H}_\sigma, u^\sigma)$ is defined by taking $\mathcal{H}_{\overline{\sigma}}$ to be the conjugate Hilbert space of $\mathcal{H}_\sigma$ and the family of projections $u^{\overline{\sigma}} = (u^{\overline{\sigma}}_{ij})$ determined by $u^{\overline{\sigma}}_{ij}(\overline{\xi}) = \overline{u^\sigma_{ji}(\xi)}$ for $\xi \in \mathcal{H}_\sigma$.

Upshot

Quantum permutations of a set I form naturally a (concrete) C^*-tensor category.

Quantum permutation groups

The collection of all finite dimensional quantum permutations of a set I forms a *discrete quantum group* $\mathrm{Sym}^+(I)$.

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The "elements" of $\mathrm{Sym}^+(I)$ are the (isomorphism classes of) irreducible finite dimensional quantum permutations of I.

The "group law" is given by taking tensor products of quantum permutations.

The "inverse" is given by taking the contragredient.

Remark

The discrete quantum group $\mathrm{Sym}^+(I)$ *is associated to the concrete rigid* C^**-tensor category of all finite dimensional quantum permutations of* I *via Tannaka-Krein reconstruction.*

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If the set $I = \{1, \ldots, n\}$ is finite there is another version of quantum permutation groups.

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Definition (Wang 1998)

The quantum permutation group S_n^+ is the universal unital C^*-algebra $C(S_n^+)$ generated by elements u_{ij} for $1 \leq i, j \leq n$ satisfying the relations of a quantum permutation.

This is naturally a *compact quantum group*.
If we abbreviate $\mathrm{Sym}^+(I) = \mathrm{Sym}_n^+$ then Sym_n^+ is *not* isomorphic to Wang's quantum permutation group S_n^+ as soon as $n \geq 4$.
However, $C(S_n^+)$ and $c_0(\mathrm{Sym}_n^+)$ have the same finite dimensional $*$-representations.

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However, $C(S_n^+)$ and $c_0(\mathrm{Sym}_n^+)$ have the same finite dimensional $*$-representations.
In fact, Sym_n^+ is the *discretization* of S_n^+ in the following sense.

Definition (Sołtan)

Let G be a compact quantum group. Then the discretization of G is the discrete quantum group G_δ associated to the concrete (rigid) C^*-tensor category of finite dimensional unital $*$-representations of the universal C^*-algebra $C(G)$ of G.

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For $|I| \geq 4$ the quantum permutation group $\mathrm{Sym}^+(I)$ is non-amenable.

Proof.

It suffices to consider the case $|I| = 4$. Banica-Bichon constructed a *matrix model* for S_4^+, giving an injective $*$-homomorphism $C(S_4^+) \to M_4(C(SO(3)))$. Using the fact that $SO(3)$ contains free subgroups one can cook up a finitely generated quantum subgroup Γ of Sym_4^+ and apply Kyed's Følner criterion to show that Γ is non-amenable. $\square$

Quantum automorphism groups of graphs

Quantum automorphism groups of graphs

In the same way as for finite graphs one can define and study quantum automorphisms of *infinite* graphs.

In the sequel we still assume that our graphs $X = (I_X, E_X)$ are simple, undirected, and without self-loops, but the sets I_X, E_X are allowed to be infinite.

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Definition

A quantum automorphism of a graph $X = (I_X, E_X)$ is the same thing as a quantum permutation $\sigma = (\mathcal{H}, u)$ of I_X such that

$$A_X u = u A_X$$

as matrices in $M_{I_X}(B(\mathcal{H}))$, where A_X is the adjacency matrix of X.

One also obtains naturally a *discrete quantum automorphism group* $\mathrm{Qut}_\delta(X)$ in this situation.

If X is finite then

$$\mathrm{Qut}_\delta(X) = \mathrm{Qut}(X)_\delta$$

is the discretization of Banica's quantum automorphism group $\mathrm{Qut}(X)$ of X.

Graphs with no quantum symmetry

The *infinite Johnson graph* $J(\infty, k)$ is the graph with vertices given by all k-element subsets of $\mathbb{N}$, such that two vertices are connected by an edge iff their intersection contains $k-1$ elements.

This graph has diameter k and is distance transitive.

Proposition (V. 2022)

The Johnson graph $J(\infty, 2)$ has no quantum symmetry.

The proof is an easy adaption of a corresponding result for finite Johnson graphs due to Schmidt.

In fact, many of the criteria and techniques developed by Schmidt carry over to the infinite setting.

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The following is again an easy generalisation of a result due to Schmidt.

Proposition

If X admits a pair of disjoint automorphisms then X has quantum symmetry.

This result allows one to give a range of examples of graphs with quantum symmetry.

Disjoint unions

Disjoint unions

Let $(X_j)_{j\in J}$ be a collection of graphs labelled by some index set J and write $X = \bigcup_{j\in J} X_j$ for their disjoint union, so that $V_X = \bigcup_{j\in J} I_{X_j}$ and $E_X = \bigcup_{j\in J} E_{X_i}$.

Theorem (V. 2022)

Let X be a connected graph. Then there is a canonical isomorphism

$$\mathrm{Qut}_\delta\left(\bigcup_{j\in J} X\right) \cong \mathrm{Qut}_\delta(X)\,\mathrm{Wr}^*\,\mathrm{Sym}^+(J)$$

of discrete quantum groups.

Here the *unrestricted free wreath product* $\Gamma\,\mathrm{Wr}^*\,\mathrm{Sym}^+(J)$ for a discrete quantum group Γ and a set J is constructed from a suitable C^*-tensor category.

If $\Gamma = G_\delta$ for a compact quantum group G and $J = \{1,\dots,n\}$ then

$$(G \wr^* S_n^+)_\delta \cong G_\delta\,\mathrm{Wr}^*\,\mathrm{Sym}_n^+$$

where $G \wr^* S_n^+$ is the free wreath product defined by Bichon.

Unit distance graphs

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A *unit distance graph* is a graph obtained by taking a subset of $\mathbb{R}^d$ as vertex set and connecting two vertices iff their Euclidean distance is equal to 1.

Examples of finite unit distance graphs in the plane include cycle graphs, hypercube graphs, and the Petersen graph.

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Consider the unit distance graph U_d associated to $\mathbb{R}^d$.

Proposition (V. 2022)

The quantum automorphism group $\mathrm{Qut}_\delta(U_1)$ *is isomorphic to the free wreath product* $\mathrm{Aut}(L)\,\mathrm{Wr}^*\,\mathrm{Sym}(\mathbb{R}/\mathbb{Z})$, *where* L *is the "infinite line" graph, i.e. the Cayley graph of* $\mathbb{Z}$ *with respect to the standard generating set* $\{\pm 1\}$.

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Question

Does U_d for $d \geq 2$ have quantum symmetry?

The Rado graph

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Definition

The Rado graph is the graph R with vertex set $I_R = \mathbb{N}$ such that a pair of vertices (m, n) is an edge iff $m < n$ and the m-th digit in the binary expansion of n is odd, or the same with the roles of m and n reversed.

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Equivalently, it can be described as the graph with vertices given by the prime numbers congruent 1 mod 4, and p, q connected iff p is a quadratic residue modulo q.

R is also known as the *random graph* since it is obtained with probability 1 by assigning edges to pairs of elements in a countable set at random.

The Rado graph

Definition

The Rado graph is the graph R with vertex set $I_R = \mathbb{N}$ such that a pair of vertices (m, n) is an edge iff $m < n$ and the m-th digit in the binary expansion of n is odd, or the same with the roles of m and n reversed.

Equivalently, it can be described as the graph with vertices given by the prime numbers congruent 1 mod 4, and p, q connected iff p is a quadratic residue modulo q.

R is also known as the *random graph* since it is obtained with probability 1 by assigning edges to pairs of elements in a countable set at random.

Key property of R

For any pair of disjoint finite sets A, B of vertices in the Rado graph R there exists a vertex w in R outside $A \cup B$ such that $(x, w) \in E_R$ for all $x \in A$ and $(y, w) \notin E_R$ for all $y \in B$.

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- R contains *all* countable graphs as induced subgraphs.
- If one removes a finite number of vertices (and all adjacent edges) or a finite number edges from R the resulting graph is again isomorphic to R.
- The set of vertices of R can be split into infinitely many disjoint sets such that the induced subgraphs are isomorphic to R.

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Question

Does the Rado graph have quantum symmetry?