Probabilistic Operator Algebra Seminar

The Infinitesimal Law of a Real Wishart Matrix

Joint work with James A. Mingo

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The moments of a real Wishart matrix are given by

$$m_n = \lim_{N\to\infty} \mathrm{E}(\mathrm{tr}(X^n)) = \sum_{\pi\in NC(n)} c^{\#(\pi)}$$

The infinitesimal moments of real Wishart matrix are given by

$$\begin{aligned} m_n' &= \lim_{N\to\infty} N\Big(\mathrm{E}(\mathrm{tr}(X^n)) - \sum_{\pi\in NC(n)} c^{\#(\pi)}\Big) \\ &= \sum_{\pi\in NC(n)} c'\#(\pi)c^{\#(\pi)-1} + \sum_{\pi\in S_{NC}^{\delta}(n,-n)} c^{\#(\pi)/2} \end{aligned}$$

The moment generating function of a real Wishart matrix is given by

$$M(z) = \sum_{n=0}^{\infty} m_n z^n = \frac{1 + (1-c)z - \sqrt{(1-az)(1-bz)}}{2z}$$

where $a = (1-\sqrt{c})^2$ and $b = (1+\sqrt{c})^2$. The corresponding Cauchy transform is

$$G(z) = \frac{1}{z} M\left(\frac{1}{z}\right) = \frac{z + 1 - c - \sqrt{(z-a)(z-b)}}{2z}.$$

We would like to compute the infinitesimal moment generating function

$$m'(z) = \sum_{n=0}^{\infty} m'_n z^n,$$

where

$$m'_n = \sum_{\pi \in NC(n)} c' \#(\pi) c^{\#(\pi)-1} + \sum_{\pi \in S_{NC}^{\delta}(n,-n)} c^{\#(\pi)/2},$$

and the infinitesimal Cauchy transform

$$g(z) = \frac{1}{z} m'\left(\frac{1}{z}\right).$$

To compute $m'(z) = \sum_{n=0}^{\infty} m'_n z^n$, let us take

$$\bar{\bar{m}}'_n = \sum_{\pi \in NC(n)} c' \#(\pi) c^{\#(\pi)-1} \qquad \text{and} \qquad \bar{m}'_n = \sum_{\pi \in S_{NC}^{\delta}(n,-n)} c^{\#(\pi)/2}$$

and consider the moment generating functions

$$\bar{\bar{m}}(z) = \sum_{n=0}^{\infty} \bar{\bar{m}}'_n z^n \qquad \text{and} \qquad \bar{m}(z) = \sum_{n=0}^{\infty} \bar{m}'_n z^n.$$

Thus, we have

$$m'(z) = \bar{\bar{m}}(z) + \bar{m}(z).$$

Note $0 = m'_0 = \bar{\bar{m}}'_0 = \bar{m}'_0 = \bar{m}'_1$.

Theorem (Mingo, 2019)

The function $\bar{\bar{g}}(z) = \sum_{n=0}^{\infty} \frac{\bar{\bar{m}}'_n}{z^{n+1}}$ *is given by*

$$\bar{\bar{g}}(z) = \frac{-c'}{z\sqrt{P(z)}} \frac{(1-c)^2 - (1+c)z - (1-c)\sqrt{P(z)}}{\sqrt{P(z)} + z - 1 + c}$$

where $P(z) = (z-a)(z-b)$ *with* $a = (1-\sqrt{c})^2$ *and* $b = (1+\sqrt{c})^2$*. Moreover, letting*

$$d\bar{\bar{\nu}}(x) = -c' \begin{cases} \delta_0 - \frac{x+1-c}{2\pi x\sqrt{(b-x)(x-a)}}\, dx & c < 1 \\ \frac{1}{2}\delta_0 - \frac{1}{2\pi\sqrt{x(4-x)}}\, dx & c = 1 \\ \quad - \frac{x+1-c}{2\pi x\sqrt{(b-x)(x-a)}}\, dx & c > 1 \end{cases},$$

we obtain

$$\bar{\bar{m}}'_n = \int x^n d d\bar{\bar{\nu}}(x)$$

Taking $m_0 = 1$, one can show that

$$m_n = (c-1)m_{n-1} + \sum_{k=1}^{n} m_{k-1}m_{n-k} \qquad \text{for} \qquad n \geq 1$$

The above recursion formula yields

$$\begin{aligned} M(z) - 1 &= \sum_{n=1}^{\infty}(c-1)m_{n-1}z^n + \sum_{n=1}^{\infty}\sum_{k=1}^{n} m_{k-1}m_{n-k}z^n \\ &= (c-1)zM(z) + z\sum_{k=1}^{\infty} m_{k-1}z^{k-1}\sum_{n=k}^{\infty} m_{n-k}z^{n-k} \\ &= (c-1)zM(z) + zM(z)M(z) \end{aligned}$$

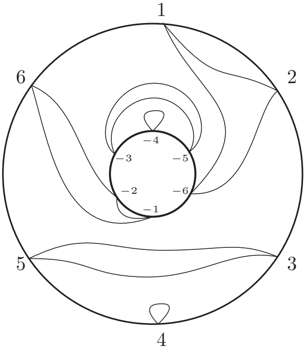
If we let $NC_k(n) = \{\pi \in NC(n) \mid \pi^{-1}(1) = k\}$ for $k = 1, 2, \ldots, n$, then

1. $NC_1(n) \simeq \{\{1\}\} \times NC(n-1)$
2. $NC_k(n) \simeq NC(k-1) \times NC(n-k)$ for $k = 2, 3, \ldots, n-1$
3. $NC_n(n) \simeq NC(n-1)$.

Thus, we get

$$
\begin{aligned}
m_n &= \sum_{\pi \in NC(n)} c^{\#(\pi)} = \sum_{\pi \in NC_1(n)} c^{\#(\pi)} + \sum_{k=2}^{n-1} \sum_{\pi \in NC_k(n)} c^{\#(\pi)} + \sum_{\pi \in NC_n(n)} c^{\#(\pi)} \\
&= c \sum_{\pi \in NC(n-1)} c^{\#(\pi)} + \sum_{k=2}^{n-1} \sum_{\alpha \in NC(k-1)} c^{\#(\alpha)} \sum_{\beta \in NC(n-k)} c^{\#(\beta)} + \sum_{\pi \in NC(n-1)} c^{\#(\pi)} \\
&= c m_{n-1} + \sum_{k=2}^{n-1} m_{k-1} m_{n-k} + m_{n-1} + m_{n-1} - m_{n-1} \\
&= (c-1) m_{n-1} + \sum_{k=1}^{n} m_{k-1} m_{n-k}
\end{aligned}
$$

$$S^{\delta}_{NC}(n,-n) = \{\sigma \in S_{NC}(n,-n) \mid \sigma\delta \text{ is a pairing }\}$$

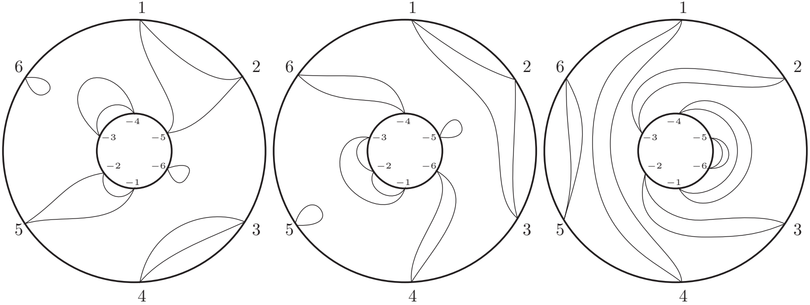


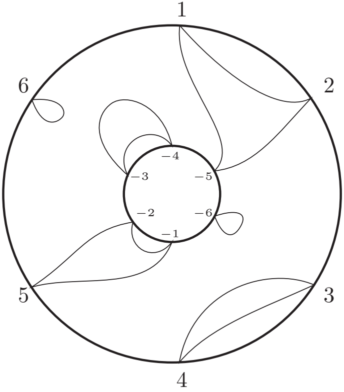
For elements of $S^{\delta}_{NC}(n,-n)$, is equivalent to requiring that $\delta\sigma\delta^{-1} = \sigma^{-1}$ and that no cycle of σ can contain both k and $-k$ for any $k \in [n]$.

$S_{NC}^{\delta}(n,-n)$ as disjoint union based on $\pi^{-1}(1)$

1. $S_I = \{\pi \in S_{NC}^{\delta}(n,-n) \mid \pi^{-1}(1) \in [-n]\}$;
2. $S_{II} = \{\pi \in S_{NC}^{\delta}(n,-n) \mid \pi^{-1}(1) \in [n]$ and the interval $I_1 = [1, \pi^{-1}(1)]$ does **not** meet a through cycle of $\pi\}$;
3. $S_{III} = \{\pi \in S_{NC}^{\delta}(n,-n) \mid \pi^{-1}(1) \in [n]$ and the interval $I_1 = [1, \pi^{-1}(1)]$ does meet a through cycle of $\pi\}$.

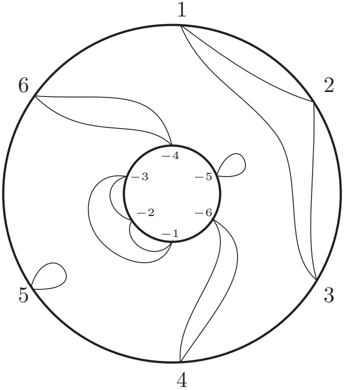
Examples of elements in S_I, S_{II}, S_{III}





$$\sum_{\pi \in S_I} c^{\#(\pi)/2} = (n-1) \sum_{\pi \in NC(n-1)} c^{\#(\pi)} = (n-1) m_{n-1}$$

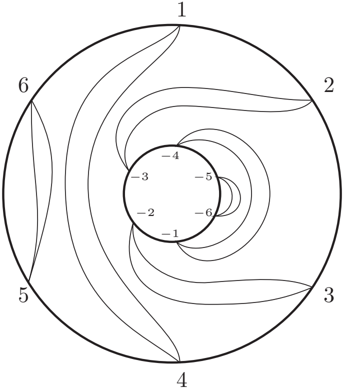
$S_{II,k} \simeq S_{NC}^{\delta}(n-k) \times NC_k(k)$ for $k = 1, 2, \ldots n-1$



$$\sum_{\pi \in S_{II}} c^{\#(\pi)/2} = c \sum_{\alpha \in S_{NC}^{\delta}(n-1)} c^{\#(\alpha)/2} + \sum_{k=2}^{n-1} \sum_{\beta \in NC(k-1)} c^{\#(\beta)} \sum_{\alpha \in S_{NC}^{\delta}(n-k)} c^{\#(\alpha)/2}$$

$$= c\overline{m}'_{n-1} + \sum_{k=2}^{n-1} m_{k-1} \overline{m}'_{n-k}$$

$S_{III,k} \simeq S_{NC,k}^{\delta}(k) \times NC(n-k)$ for $k = 2, 3, \dots n$



$$\sum_{\pi \in S_{III}} c^{\#(\pi)/2} = \sum_{\alpha \in S_{NC}^{\delta}(n-1)} c^{\#(\alpha)/2} + \sum_{k=2}^{n-1} \sum_{\beta \in NC(n-k)} c^{\#(\beta)} \sum_{\alpha \in S_{NC}^{\delta}(k-1)} c^{\#(\alpha)/2}$$

$$= \overline{m}'_{n-1} + \sum_{k=2}^{n-1} m_{n-k} \overline{m}'_{k-1}$$

From the above we get

$$\begin{aligned}\overline{m}'_n &= \{(n-1)m_{n-1}\} + \left\{ c\overline{m}'_{n-1} + \sum_{k=2}^{n-1} m_{k-1}\overline{m}'_{n-k} \right\} + \left\{ \overline{m}'_{n-1} + \sum_{k=2}^{n-1} m_{n-k}\overline{m}'_{k-1} \right\} \\ &= (n-1)m_{n-1} + (c-1)\overline{m}'_{n-1} + \sum_{k=1}^{n-1} m_{k-1}\overline{m}'_{n-k} + \sum_{k=2}^{n} m_{n-k}\overline{m}'_{k-1}\end{aligned}$$

Recall that $0 = \overline{m}'_0 = \overline{m}'_1$. Thus, we have

$$\begin{aligned}\overline{m}'_n &= (n-1)m_{n-1} + (c-1)\overline{m}'_{n-1} + \sum_{k=1}^{n} \{m_{k-1}\overline{m}'_{n-k} + m_{n-k}\overline{m}'_{k-1}\} \\ &= (n-1)m_{n-1} + (c-1)\overline{m}'_{n-1} + 2\sum_{k=1}^{n} m_{k-1}\overline{m}'_{n-k}\end{aligned}$$

Letting

$$\overline{m}'_0 = 0 \quad \text{and} \quad \overline{m}'_n = \sum_{\pi \in S^{\delta}_{NC}(n,-n)} c^{\#(\pi)/2} \quad \text{for} \quad n \geq 1,$$

we want to compute

$$\overline{m}(z) = \sum_{n=0}^{\infty} \overline{m}'_n z^n . \frac{z^2 M'(z)}{1 + z(1-c) - 2z, M(z)}$$

Since we proved that

$$\overline{m}_n' = (n-1)m_{n-1} + (c-1)\overline{m}_{n-1}' + 2\sum_{k=1}^{n} m_{k-1}\overline{m}_{n-k}',$$

We obtain

$$\overline{m}(z) = \sum_{n=1}^{\infty}\left\{(n-1)m_{n-1} + (c-1)\overline{m}_{n-1}' + 2\sum_{k=1}^{n} m_{k-1}\overline{m}_{n-k}'\right\} z^n.$$

Similar calculations to those for $M(z)$ yield

$$\sum_{n=1}^{\infty}(n-1)m_{n-1}z^n = z^2M'(z), \qquad \sum_{n=1}^{\infty}\overline{m}_{n-1}'z^n = z\overline{m}(z), \qquad \text{and}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} m_{k-1}\overline{m}_{n-k}'z^n = z\overline{m}(z)M(z)$$

$$\sum_{n=1}^{\infty}(n-1)m_{n-1}z^n = z^2M'(z), \qquad \sum_{n=1}^{\infty}\overline{m}'_{n-1}z^n = z\overline{m}(z), \qquad \text{and}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} m_{k-1}\overline{m}'_{n-k}z^n = z\overline{m}(z)M(z)$$

Theorem (Mingo, V., 2022)

The infinitesimal moment generating function $\overline{m}(z) = \sum_{n=0}^{\infty}\overline{m}'_n z^n$ *satisfies the equation*

$$\overline{m}(z) = z^2M'(z) + (c-1)z\overline{m}(z) + 2z\overline{m}(z)M(z)$$

with solution

$$\overline{m}(z) = \frac{z^2M'(z)}{1 + z(1-c) - 2zM(z)}$$

Now, for the transform $\bar{g}(z) = \frac{1}{z}\overline{m}\left(\frac{1}{z}\right)$, we get

$$\bar{g}(z) = \frac{1}{z}\overline{m}\left(z^{-1}\right) = \frac{1}{z}\frac{z^{-2}M'(z^{-1})}{1 + z^{-1}(1-c) - 2z^{-1}M(z^{-1})}$$

Finally, considering the relations

- $M(z^{-1}) = zG(z)$
- $-z^{-2}M'(z^{-1}) = G(z) + zG'(z)$
- $G(z) + zG'(z) = \dfrac{1 - zG(z)}{\sqrt{P(z)}}$ where $P(z) = (z-a)(z-b)$ with $a = (1-\sqrt{c})^2$ and $b = (1+\sqrt{c})^2$

we get

Theorem (Mingo, V., 2022)

$$\bar{g}(z) = \frac{zG(z) - 1}{P(z)} = \frac{1}{2}\left\{\frac{1}{2}\left\{\frac{1}{z-a} + \frac{1}{z-b}\right\} - \frac{1}{\sqrt{(z-a)(z-b)}}\right\}.$$

Thank you!