Categorical probability with Markov categories

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References

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  arXiv:2001.08375.

For a broader perspective, see the videos from the online workshop Categorical Probability and Statistics!

The big picture

Traditional probability theory	Categorical probability theory
Analytic: says what probabilities are	Synthetic: says how probabilities behave
Analogous to number systems	Analogous to abstract algebra

- Compare with:

$$\text{traditional homotopy theory} \iff \text{homotopy type theory}$$

- There will be no numerical probabilities and no measure theory!

Results so far

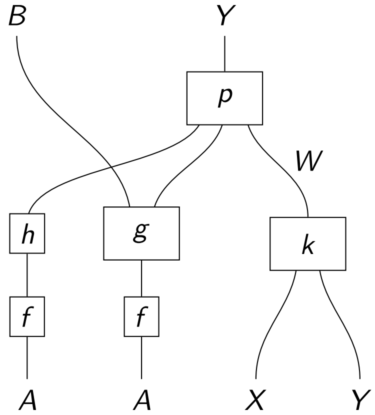
Categorical proofs and generalizations of a number of classical results:

- Theorems on sufficient statistics (Fisher–Neyman's, Basu's, Bahadur's).
- 0/1-laws (Kolmogorov's, Hewitt–Savage's).
- Blackwell–Sherman–Stein theorem on comparison of statistical experiments.
- **De Finetti theorem**.

The basic primitives are morphisms in a symmetric monoidal category:

- **Intuitively**, a morphism is a probabilistic function: random output on given input.
- We impose axioms that (partly) formalize this intuition.

We can compose morphisms using string diagram calculus, like this:

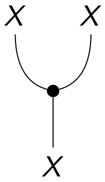


This defines an overall morphism

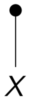
$$A \otimes A \otimes X \otimes Y \longrightarrow B \otimes Y.$$

Postulate additional pieces of structure:

- Every object X has a **copying function**:

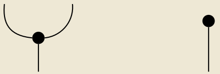


- Every object X has a **deletion function**:

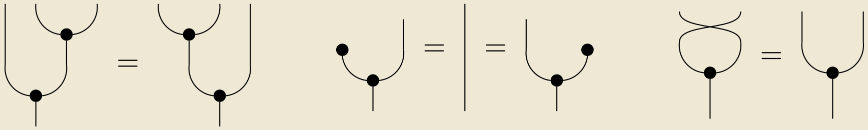


Definition

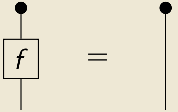
A **Markov category C** is a symmetric monoidal category supplied with **copying** and **deleting** morphisms on every object,



giving commutative comonoid structures



which interact well with the monoidal structure, and such that for all f,



Semantics

I describe *one* Markov category to convince you that Markov categories can describe probability.

BorelStoch is the category with:

- **Standard Borel spaces** as objects (finite sets, $\mathbb{N}$ and $[0, 1]$).
- Measurable **Markov kernels** as morphisms.
- Products of measurable spaces for $\otimes$.
- The obvious copying morphisms.

BorelStoch satisfies all additional axioms that I will mention.

Von Neumann algebras

There is also a Markov category with:

- **Commutative von Neumann algebras** as objects.
- Formal opposites of **normal positive unital maps** as morphisms.
- The **Dauns tensor product** for $\otimes$.
- The (formal opposite of) multiplication as the copying morphism!

Should work similarly in the non-commutative case, resulting in a **quantum Markov category** (Parzygnat).

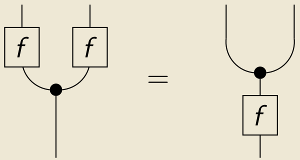
What about the free case?

Determinism

Throughout, we're in a Markov category $\mathbf{C}$.

Definition

A morphism $f : X \to Y$ is **deterministic** if it commutes with copying,



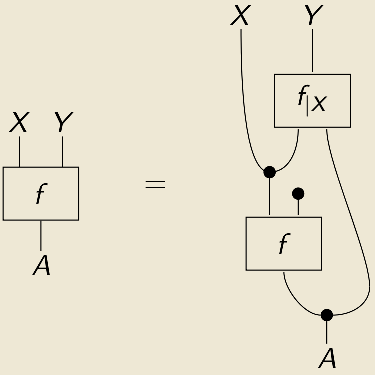
- **Intuition:** Applying f to copies of input $=$ copying the output of f.

- The deterministic morphisms form a cartesian monoidal subcategory $\mathbf{C}_{\text{det}}$.

Conditionals

Axiom

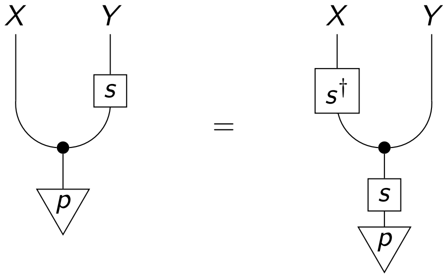
C has conditionals if for every $f : A \to X \otimes Y$ there is $f_{|X} : X \otimes A \to Y$ with



- **Intuition:** The outputs of f can be generated one at a time.

Bayesian inversion

Every $s : X \to Y$ has a **Bayesian adjoint** $s^\dagger : Y \to X$ satisfying:



The Bayesian adjoint $s^\dagger$ depends on p.

Infinite tensor products

Let $(X_i)_{i \in I}$ be a family of objects.

For finite $F \subseteq F' \subseteq I$, we have projection morphisms

$$\bigotimes_{i \in F'} X_i \longrightarrow \bigotimes_{i \in F} X_i$$

given by composing with deletion for all $i \in F' \setminus F$.

Infinite tensor products

Definition

The **infinite tensor product**

$$X^I := \bigotimes_{i \in I} X_i$$

is the limit of the finite tensor products $X^F := \bigotimes_{i \in F} X_i$ if it exists and is preserved by every $- \otimes Y$.

- **Intuition:** To map into an infinite tensor product, one needs to map consistently into its finite subproducts.
- Turns the **Kolmogorov extension theorem** into a definition.
- Reproduces (the formal opposite of) the infinite tensor products of rings.

Kolmogorov products

Definition

An infinite tensor product X^I is a **Kolmogorov product** if the limit projections $\pi^F : X^I \to X^F$ are deterministic.

- This additional condition fixes the comonoid structure on X^I.

Axiom

C has countable Kolmogorov products.

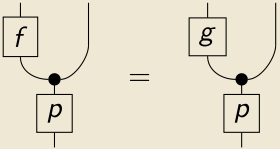
- Need this already in order to state the de Finetti theorem.

Almost sure equality

Definition

Let $p : A \to X$ and $f, g : X \to Y$.

f and g are **equal p-almost surely**, $f =_{p\text{-a.s.}} g$, if



- **Intuition:** f and g behave the same on all inputs produced by p.
- Other concepts (besides equality) also relativize with respect to p-almost surely.

Representability

Axiom

A Markov category C is **representable** if for every $X \in \mathsf{C}$ there is $PX \in \mathsf{C}$ and a natural bijection

$$\mathsf{C}_{\mathrm{det}}(-, PX) \cong \mathsf{C}(-, X),$$

and **a.s.-compatibly representable** if this respects p-$\mathrm{a.s.}$ equality for every p.

- **Intuition:** PX is space of probability measures on X.
- Under the bijection, the deterministic $\mathrm{id} : PX \to PX$ corresponds to

$$\mathsf{samp}_X : PX \to X,$$

the map that returns a random sample from a distribution.

Detour: random measures

- Suppose that I hand you a coin (which may be biased).

- How much would you bet on the outcome

$$\text{heads}, \quad \text{tails}, \quad \text{tails}$$

  when the coin is flipped 3 times?

  $\Rightarrow$ Surely the same as you would bet on

$$\text{tails}, \quad \text{tails}, \quad \text{heads}.$$

- Your bets satisfy **permutation invariance**. Can we say more?

Classical de Finetti theorem

A sequence $(x_n)_{n\in\mathbb{N}}$ of random variables on a space X is **exchangeable** if their distribution is invariant under finite permutations σ,

$$\mathbb{P}\big[x_1 \in S_{\sigma(1)}, \dots, x_n \in S_{\sigma(n)}\big]$$
$$= \mathbb{P}[x_1 \in S_1, \dots, x_n \in S_n]\,.$$

Theorem

If (x_n) is exchangeable, then there is a measure μ on PX such that

$$\mathbb{P}[x_1 \in S_1, \dots, x_n \in S_n] = \int p(x_1 \in S_1) \cdots p(x_n \in S_n)\, \mu(dp).$$

Idea: sequence of tosses of a coin with unknown bias!

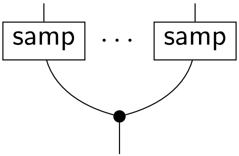
The de Finetti theorem

Assumption: All three axioms above hold. (True for **BorelStoch**.)

Definition

$p : A \to X^{\mathbb{N}}$ is **exchangeable** if it is invariant under composing with finite permutations.

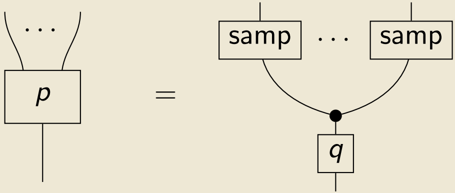
Sampling $\mathbb{N}$ times gives a morphism $PX \to X^{\mathbb{N}}$ given by



The de Finetti theorem

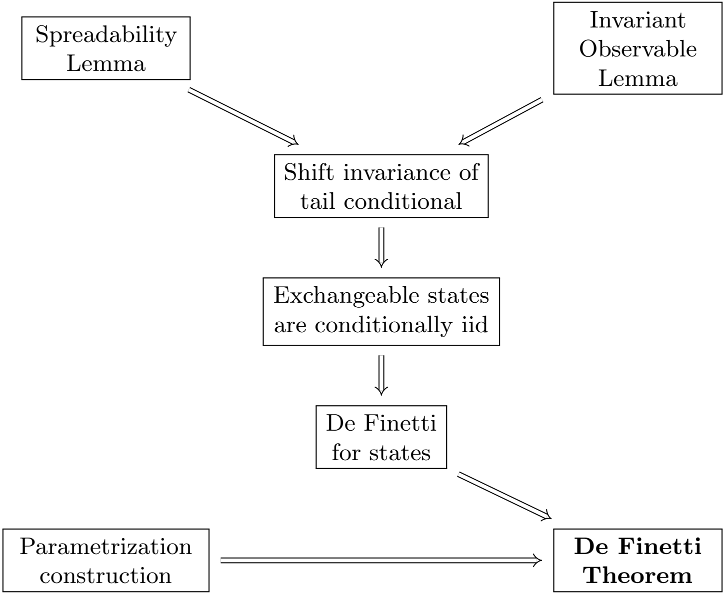
Theorem

For every exchangeable $p : A \to X^{\mathbb{N}}$ there is $q : A \to PX$ such that



- **Intuition:** The probabilities associated to your bets arise from **sampling from a random distribution**.

Structure of proof



Spreadability Lemma

Lemma

If $p : A \to X^{\mathbb{N}}$ is exchangeable, then p is also invariant with respect to applying any injective map $\mathbb{N} \to \mathbb{N}$ to the tensor factors.

- **Intuition:** If random variables $x_1, x_2, \ldots$ are permutation-invariant, then they have the same distribution as $x_2, x_3, \ldots$

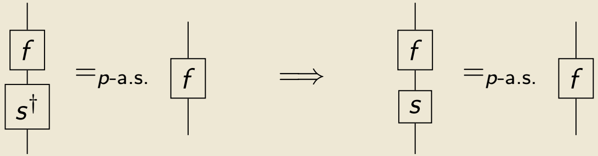
Proof sketch. On every finite $F \subseteq \mathbb{N}$, every injection $\mathbb{N} \to \mathbb{N}$ coincides with a suitable permutation.

Invariant Observable Lemma

Lemma

Let $p\colon I \to X$ and $s\colon X \to X$ satisfy $sp = p$.

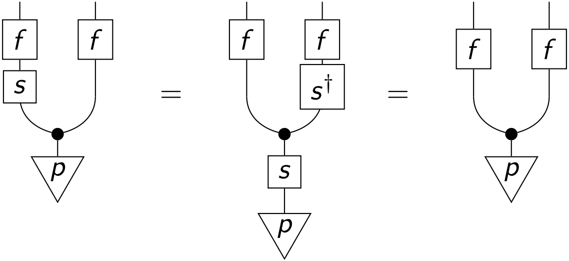
Then for deterministic $f\colon X \to Y$,



- **Intuition:** s and p make X into a **measure-preserving dynamical system**, f is an observable.
- If f is invariant "backward in time", then it is also invariant "forward in time".

Invariant Observable Lemma

Proof sketch.



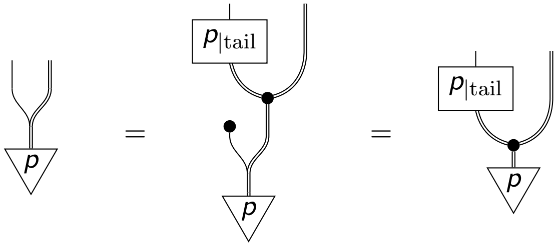
Like an equation between inner products in "$L^2(A, p)$".

$\Rightarrow$ The claim follows by "Cauchy-Schwarz". □

The tail conditional

We use double wires to denote $X^{\mathbb{N}}$.

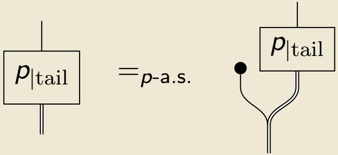
By the existence of conditionals, there is $p_{|\mathrm{tail}}$ such that



The second equation is by the Spreadability Lemma.

Shift invariance of the tail conditional

Lemma

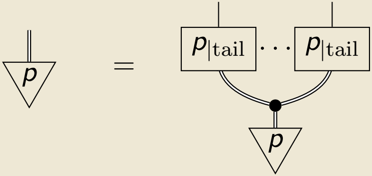


- **Intuition:** $p_{|\mathrm{tail}}$ is independent of any finite initial segment.

Proof sketch. An application of the Invariant Observable Lemma. Its assumption holds by the Spreadability Lemma. □

Exchangeable states are conditionally iid

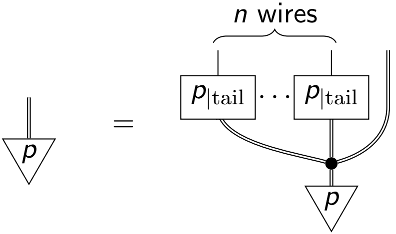
Lemma



- **Intuition:** Conditional on the tail, the outputs of p are independent.
- This is sometimes presented as “the” de Finetti theorem.
- It implies the earlier version relatively easily.

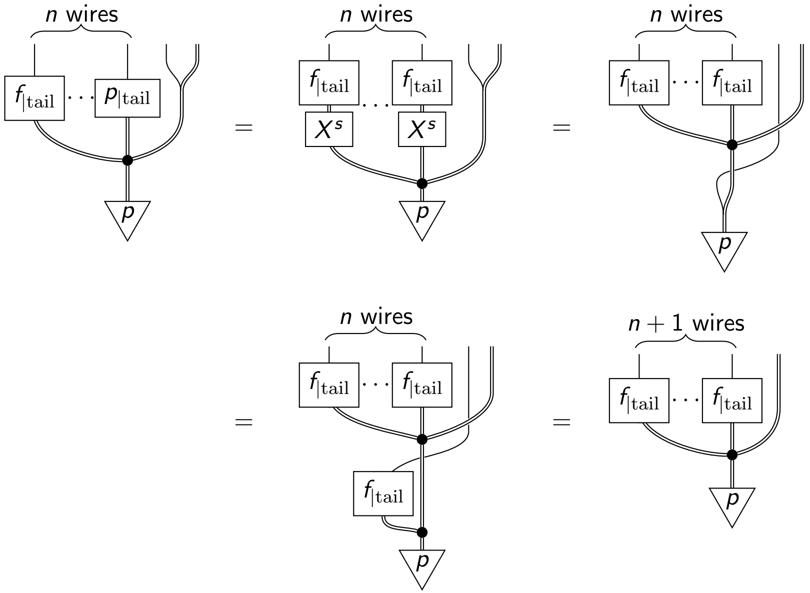
Exchangeable states are conditionally iid

Proof sketch. It is enough to show



for every finite n.

Using induction on n,



□

So, why categorical probability?

- Prove theorems in **greater generality** and with **more intuitive proofs**.
- **Reverse mathematics:** sort out interdependencies between theorems.
- Ultimately, prove **theorems of higher complexity**?
- Different **conceptual perspective** on what probability is.
- Applications to **probabilistic programming**. (And machine learning?)
- **Simpler teaching** of probability theory. (String diagrams!)

Summary and Outlook

- Markov categories are an emerging framework for "synthetic" probability theory.
- We already have abstract versions of some theorems of probability and statistics:
- Some developments require turning theorems into definitions.
- Tantalizing connections with ergodic theory.
- **Next:** a treatment of the law of large numbers.
- In parallel, we also aim at a better understanding of the axiomatics and the semantics.

Bonus slides: Kleisli categories are Markov categories

Proposition

Let

- **D** be a category with finite products,
- P a commutative monad on **D** with $P(1) \cong 1$.

Then the Kleisli category $\mathrm{Kl}(P)$ is a Markov category in the obvious way.

Examples:

- Kleisli category of the Giry monad, other related monads for measure-theoretic probability.
- Kleisli category of the non-empty power set monad, which is (almost) **Rel**.

The proposition still holds when **D** is merely a Markov category itself!

Categories of comonoids

Proposition

Let **C** be any symmetric monoidal category. Then the category with:

- Commutative comonoids in **C** as objects,
- Counital maps as morphisms,
- The specified comultiplications as copy maps,

is a Markov category.

A good example is $\mathbf{Vect}_k^{\mathrm{op}}$ for a field k:

- The comonoids correspond to commutative k-algebras of k-valued random variables.
- We obtain **algebraic probability theory** with "random variable transformers" as morphisms (formal opposites of Markov kernels).

Diagram categories and ergodic theory

Proposition

Let $\mathbf{D}$ be any category and $\mathbf{C}$ a Markov category. The category in which

- Objects are functors $\mathbf{D} \to \mathbf{C}_{\mathrm{det}}$,
- Morphisms are natural transformations with components in $\mathbf{C}$.

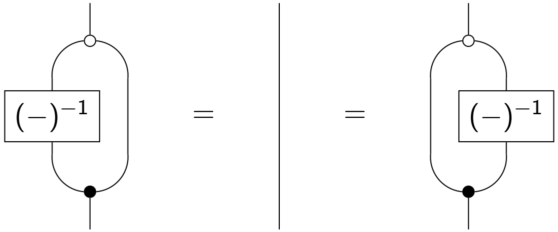
With the poset $\mathbf{D} = \mathbb{Z}$, we get a category of **discrete-time stochastic processes**.

This generalizes an observation going back to (Lawvere, 1962).

We can also take $\mathbf{D} = \mathbf{B}G$ for a group G, resulting in categories of dynamical systems with deterministic dynamics but stochastic morphisms.

Hyperstructures: categorical algebra in Markov categories

A **group** G is a monoid G together with $(-)^{-1} : G \to G$ such that



This equation can be interpreted in any Markov category! (Together with the bialgebra law.)

- More generally, one can consider models of any algebraic theory in any Markov category.

- In Kleisli categories of probability-like monads, these are known as **hyperstructures**.

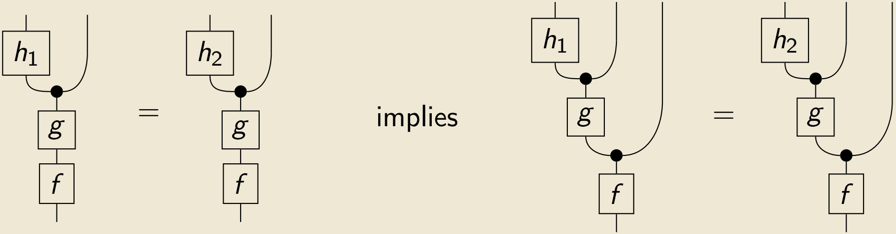
- Peter Arndt's suggestion:

  Develop categorical algebra for hyperstructures in terms of Markov categories!

The causality axiom

Definition

C is **causal** if

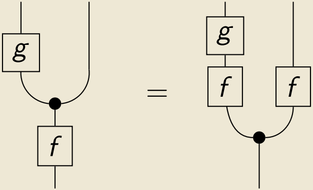


- **Intuition:** The choice between h_1 and h_2 in the "future" of g does not influence the "past" of g.
- Not every Markov category is causal.

The positivity axiom

Definition

C is **positive** if whenever gf is deterministic for composable f and g, then also



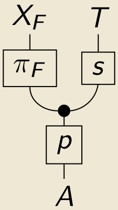
- **Intuition:** If a deterministic process has a random intermediate result, then that result can be computed independently from the process.
- Not every Markov category is positive.
- Dario Stein: every causal Markov category is positive!

Theorem (Kolmogorov zero–one law)

Let X_I be a Kolmogorov product of a family $(X_i)_{i \in I}$.

If

- $p : A \to X_I$ makes the X_i independent and identically distributed, and
- $s : X_I \to T$ is such that



  displays $X_F \perp T \,||\, A$ for every finite $F \subseteq I$,

then ps is deterministic.

The classical Hewitt–Savage zero-one law

Theorem

Let $(x_n)_{n \in \mathbb{N}}$ be independent and identically distributed random variables, and S any event depending only on the x_n and invariant under finite permutations.

Then $\boldsymbol{P}(S) \in \{0, 1\}$.

The synthetic Hewitt–Savage zero-one law

Theorem

Let J be an infinite set and $\mathbf{C}$ a causal Markov category. Suppose that:

- The Kolmogorov power $X^{\otimes J} := \lim_{F \subseteq J \text{ finite}} X^{\otimes F}$ exists.
- $p : A \to X^{\otimes J}$ displays the conditional independence $\perp_{i \in J} X_i \parallel A$.
- $s : X^J \to T$ is deterministic.
- For every finite permutation $\sigma : J \to J$, permuting the factors $\tilde{\sigma} : X^{\otimes J} \to X^{\otimes J}$ satisfies

$$\tilde{\sigma} p = p, \qquad s \tilde{\sigma} = s.$$

Then sp is deterministic.

Proof is by string diagrams, but far from trivial!

Discrete probability theory as a Markov category

FinStoch is the category of finite sets and **stochastic matrices**: a morphism $f : X \to Y$ is

$$(f(y|x))_{x \in X, y \in Y} \in \mathbb{R}^{X \times Y}$$

with

$$f(y|x) \geq 0, \qquad \sum_y f(y|x) = 1.$$

Composition is the **Chapman-Kolmogorov formula**,

$$(gf)(z|x) := \sum_y g(z|y)\, f(y|x).$$

A morphism $p : 1 \to X$ is a **probability distribution**.

A general morphism $X \to Y$ has many names: **Markov kernel**, probabilistic mapping, communication channel, ...

The monoidal structure implements **stochastic independence**,

$$(g \otimes f)(xy|ab) := g(x|a)\, f(y|b).$$

The copy maps are

$$\mathsf{copy}_X \;:\; X \longrightarrow X \times X, \qquad \mathsf{copy}_X(x_1, x_2|x) = \begin{cases} 1 & \text{if } x_1 = x_2 = x, \\ 0 & \text{otherwise.} \end{cases}$$

The deletion maps are the unique morphisms $X \to 1$.

- Works just the same with "probabilities" taking values in any **semiring** R.
- Taking R to be the **Boolean semiring** $\mathbb{B} = \{0, 1\}$ with

$$1 + 1 = 1$$

  results in the Kleisli category of the nonempty finite powerset monad.

  $\Rightarrow$ We get a Markov category for non-determinism.
- Measure-theoretic probability: Kleisli category of the **Giry monad**.