Metric Convergence of Spectral Triples on the Sierpinski Gasket and Other Piecewise C^1-fractal Curves

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February 8, 2021

Piecewise C^1-fractal Curves and Approximation Sequences

Spectral Triples

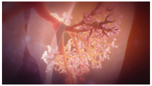
Gromov-Hausdorff Propinquity

Metric Approximations of Spectral Triples on Piecewise C^1-fractal Curves via the Spectral Propinquity

Overview

Finite Approximations of Fractals

- better understand how fractal structures arise and evolve in nature




- extend methods from mathematical physics classically formulated on smooth manifolds to fractal spaces

Tools from Noncommutative Geometry

- spectral triples
  -generalize differentiable structure
- Gromov-Hausdorff propinquity
  -extends Hausdorff distance to function spaces

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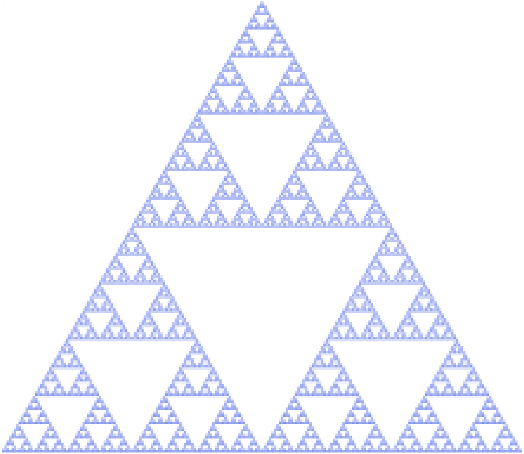
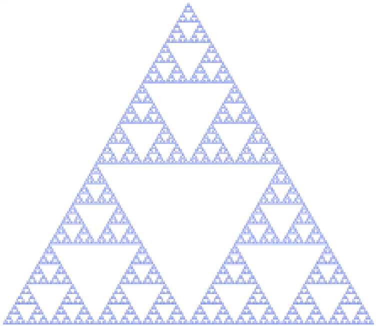
Motivating Example: The Sierpinski Gasket



Motivating Example: The Sierpinski Gasket



Let p_i denote the vertices of a regular 3-simplex, and for $i = 1, 2, 3$, let

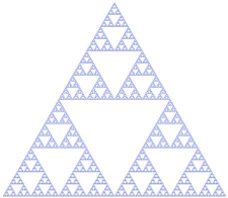
$$F_i x = \frac{1}{2}(x - p_i) + p_i.$$

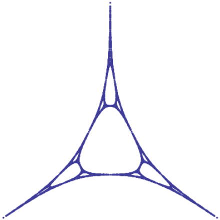
The *Sierpinski gasket* SG is the unique nonempty compact subset of $\mathbb{R}^2$ such that $SG = \bigcup_{i=1}^{3} F_i(SG)$.

Piecewise C^1-fractal Curve (Lapidus, Sarhad)

A *piecewise C^1-fractal curve* is a compact length space $X \subset \mathbb{R}^n$ that satisfies the axioms below. Let $L(\gamma)$ denote the length of the continuous curve γ parametrized by its arclength.

- Axiom 1. $X = \overline{R}$ where $R = \bigcup_{j \geq 1} R_j$ and R_j, $j \in \mathbb{N}$, is a rectifiable C^1 curve with $L(R_j) \to 0$ as $j \to \infty$.
- Axiom 2. There exists a dense subset $B \subseteq X$ such that for each $p \in B$ and $q \in X$, one of the minimizing geodesics from p to q can be given as a countable (or finite) concatenation of the R_j's.





Important Properties of Harmonic Functions on the Sierpinski Gasket

- $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$
- $V_0 = \{p_1, p_2, p_3\}$
- $V_n = \bigcup_{w_i \in \{1,2,3\}^n} F_w(V_0)$ where w is an n-dimensional vector and F_w denotes $F_{w_1} \circ \cdots \circ F_{w_n}$
- $V_* = \bigcup_{n \geq 0} V_n$
- If f and g are real-valued functions on SG_n, then the *energy on SG_n* is given by

$$E_n(f, g) = \sum_{x \sim_n y} (f(x) - f(y))(g(x) - g(y)),$$

where $x \sim_n y$ signifies that x and y are connected by a single edge in SG_n. An extension of f to V_{n+1} that minimizes $E_{n+1}(f) := E_{n+1}(f, f)$ is defined as the *harmonic extension of f to V_{n+1}*. A function $f : V_n \to \mathbb{R}$ which, given its values at V_0, minimizes $E_k(f)$ for each $k = 1, 2, \ldots, n$ is called a *harmonic function*.

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Important Properties of Harmonic Functions on the Sierpinski Gasket

- A real-valued function on V_0 can be uniquely extended to a real-valued, harmonic function on SG.
- (Kigami) Let $E(f;g) := \lim_{n \to \infty} \left(\frac{5}{3}\right)^n E_n(f;g)$, be a probability measure on SG, $f \in \mathrm{dom}(E)$, and $u \in C(SG)$. Note that $g \in \mathrm{dom}(E)$ if $E(g;g) < \infty$. Consider , *the Laplacian on SG with respect to the measure* , i.e. $f = u$ if

$$E(f;g) = \int_{SG} ug \, d$$

for all $g \in \mathrm{dom}(E)$. If f is harmonic, then $f \in \mathrm{dom}(\;)$ and $f = 0$. Conversely, if $f \in \mathrm{dom}(\;)$ and $f = 0$, then f is harmonic.

Construction of the Harmonic Gasket

- $V_0 = \{p_1, p_2, p_3\}$
- for each $j = 1, 2, 3,$ let the function $h_j : V_0 \to \mathbb{R}^3$ be given by $h_j(p_k) = \ _j(k)$ for $k = 1, 2, 3$
- extend h_j harmonically to V and by continuity to SG
- $: SG \to \mathbb{R}^3$ is defined by

$$(x) = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \end{pmatrix} \quad \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

- the harmonic gasket, K_H, can be given by $K_H := \ (SG)$

Approximating Piecewise C^1-fractal Curves

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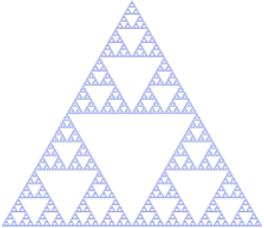

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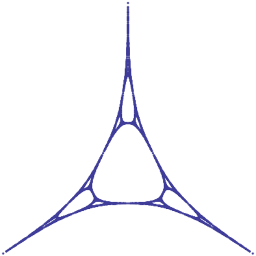

Approximation Sequence for a Piecewise C^1-fractal Curve Compatible with a Given Parameterization (L., Lapidus, Latrémolière)

Let X be a piecewise C^1-fractal curve with parameterization $(R_j)_{j \in \mathbb{N}}$. An approximation sequence of X compatible with $(R_j)_{j \in \mathbb{N}}$ is a strictly increasing function $B : \mathbb{N} \to \mathbb{N}$ such that, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, and letting

- $X_n = \bigcup_{j=1}^{B(n)} R_j$,
- V_n denote the set of the endpoints of the curves $R_1, \ldots, R_{B(n)}$,
- d_n be the geodesic distance on X_n,

the following properties hold:

(1) $\mathrm{Haus}_{d_n}(V_n, X_n) < \varepsilon$,

(2) the restriction of d_1 to $V_n \times V_n$ is d_n.

Approximation Sequence for the Sierpinski Gasket

Let R_j, denote, for each $j \geq 1$, a continuous, injective functions to edges in SG_n such that

$$R_j : [0,1] \to \text{ the edges in } SG_0 \text{ for } j = 1,2,3;$$
$$R_j : [0,2^{-1}] \to \text{ the edges in } SG_1 \text{ for } j = 4,5,\ldots,12$$
$$R_j : [0,2^{-2}] \to \text{ the edges in } SG_2 \text{ for } j = 13,14,\ldots,39$$
$$\vdots$$
$$R_j : [0,2^{-n}] \to \text{ the edges in } SG_n \text{ for } j = 1+\sum_{i=1}^{n} 3^i, 2+\sum_{i=1}^{n} 3^i, \ldots, 3^{n+1}+\sum_{i=1}^{n} 3^i$$

Let $B : \mathbb{N} \to \mathbb{N}$ be given by $B(n) = \sum_{i=1}^{n+1} 3^i$. Then $B(n)$ defines an approximation sequence of SG compatible with the parameterization $(R_j)_{j \in \mathbb{N}}$.

Approximation Sequence for the Harmonic Gasket

Let $B : \mathbb{N} \to \mathbb{N}$ be given by $B(n) = \sum_{i=1}^{n+1} 3^i$. Then $B(n)$ defines an approximation sequence of HG compatible with the parameterization $(R_j)_{j \in \mathbb{N}}$.

Spectral Triple (Connes)

Let A be a unital C -algebra. An unbounded Fredholm module $(H; D)$ over A consists of a Hilbert space H together with a unital representation of A into $B(H)$ and an unbounded, self-adjoint operator D on H such that

- the set

  $$f\, a \in A \text{ for which } [D; \ (a)] \text{ is densely de ned}$$

  $$\text{and extends to a bounded operator on } H g$$

  is dense in A
- the operator $(I + D^2)^{\ 1}$ is compact.

If the underlying representation is faithful, then $(A; H; D)$ is called a *spectral triple*, and D a Dirac operator.

Theorem (Lapidus, Sarhad 2014)

Let X be a compact length space that satisfies Axioms 1 and 2. Then $X = \overline{\bigcup_{j \geq 1} R_j}$, where R_j is a rectifiable C^1 curve of length l_j for each $j \in \mathbb{N}$. Set

- $H_{l_j} := L^2([-l_j, l_j], (2l_j)^{-1} m)$
- $D_{l_j} := D^0_{l_j} + \frac{\pi}{2l_j} I$, where $D^0_{l_j} = \overline{-i\frac{d}{dx}|_{\mathrm{span}(\phi^{l_j}_k)}}$ and $\phi^{l_j}_k = \exp(\frac{i\pi kx}{l_j})$, $k \in \mathbb{Z}$
- $\pi_{l_j}(f)h(x) := f((R_j(|x|)))h(x)$ for $f \in C(X)$, $h \in H_{l_j}$

Then $(C(X), \bigoplus_{j \geq 1} H_{l_j}, \bigoplus_{j \geq 1} D_{l_j})$ with $\pi = \bigoplus_{j \geq 1} \pi_{l_j}$ is a spectral triple for X that recovers $d_\infty(x, y)$ via

$\sup\{|f(x) - f(y)| : f \in C(X), \|[D_\infty, \pi_\infty(f)]\|_{B(H_\infty)} \leq 1\}$.

Recovery of Geodesic Distance

$$d_1(p;q) = \inf\{ L(\ ) : \ \text{is a path from } p \text{ to } q \} \text{ where}$$

$$L(\ ) = \int_p^q (g \ \ dx \ \ dx \ )^{\frac{1}{2}}$$

is dual to

$d_1(p;q)$

$$= \sup\{ | f(p) \quad f(q) | : f \in C(X); \| [D_1 ; \ (f)] \|_{B(H_1)} \quad 1 \}$$

Extending Hausdor Distance to Function Spaces

!

$$SG_n \ !^{Haus_{d_{eucl}}} \ SG$$

$$(SG_n; d_n) !^{GH} (SG; d_1)$$

$$(C(SG_n); L_{d_n}) !^{?} (C(SG); L_{d_1})$$

Quantum Compact Metric Space (Rieffel)

Let A be a unital C^*-algebra. The *state space* $S(A)$ of A is the set of positive linear functionals on A of norm 1. If L is a seminorm defined on a dense subspace of the self-adjoint elements of A satisfying some form of Leibniz inequality and such that

$$\{a \in sa(A) : L(a) = 0\} = \mathbb{R}1_A$$

and the associated *Monge-Kantorovich distance*, that is, the metric defined for all $\varphi, \psi \in S(A)$ by

$$mk_L(\varphi; \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in dom(L); L(a) \leq 1\};$$

metrizes the weak* topology of $S(A)$, then $(A; L)$ is a quantum compact metric space $(A; L)$.

Metric Spectral Triples (Latrémolière)

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. Given $a \in \mathcal{A}$, set $L_D(a) = \|[D, \pi(a)]\|_{B(\mathcal{H})}$. If $(\mathcal{A}, L_D)$ is a quantum compact metric space, then $(\mathcal{A}, \mathcal{H}, D)$ is a *metric spectral triple*.

The *spectral propinquity* is a metric on the class of metric spectral triples.

Theorem (Latrémolière 2018)

If $(A; H; D)$ and (A', H', D') are metric spectral triples with spectral propinquity $^{spec}((A; H; D); (A', H', D')) = 0$, then there exists a unitary $U : H \to H'$ and a *-isomorphism $: A \to A'$ such that

$$UDU = D';$$

and for every $a \in A$ and $\in H'$,

$$(a) = (UaU):$$

Note that is also a full quantum isometry- that is, $L_{D'} = L_D$.

Spectral Propinquity Metric Convergence of Spectral Triples on SG (L., Lapidus, Latrémolière)

Let $\{R_j\}_{j\in\mathbb{N}}$ be a parameterization of SG as a piecewise C^1-fractal curve and $B(n)$ an approximation sequence of SG compatible with $\{R_j\}_{j\in\mathbb{N}}$. Denote the Lapidus and Sarhad spectral triple on SG, $(C(SG), \bigoplus_{j\geq 1} H_{l_j}, \bigoplus_{j \geq 1} \slashed{D}_{l_j})$, by $(C(SG), H_\infty, \slashed{D}_\infty)$ and $(C(SG_{B(n)}), \bigoplus_{j=1}^{B(n)} H_{l_j}, \bigoplus_{j=1}^{B(n)} \slashed{D}_{l_j})$, by $(C(SG_{B(n)}), H_{B(n)}, \slashed{D}_{B(n)})$.

When equipped with $L_{\slashed{D}_\infty}(a) := \|[\slashed{D}_\infty, \pi(a)]\|_{B(H_\infty)}$, $(C(SG), L_{\slashed{D}_\infty})$ is a quantum compact metric space. Similarly, $(C(SG_{B(n)}), L_{\slashed{D}_{B(n)}})$ is also a quantum compact metric space.

Moreover,

$$\lim_{n\to\infty} \Lambda^{spec}((C(SG_{B(n)}), H_{B(n)}, \slashed{D}_{B(n)}), (C(SG), H_\infty, \slashed{D}_\infty)) = 0.$$

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When equipped with $L_{\not{D}_\infty}(a) := \|[\not{D}_\infty, \pi(a)]\|_{\mathcal{B}(\mathcal{H}_\infty)}$, $(C(SG), L_{\not{D}_\infty})$ is a quantum compact metric space. Similarly, $(C(SG_{B(n)}), L_{\not{D}_{B(n)}})$ is also a quantum compact metric space.

Moreover,

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}((C(SG_{B(n)}), \mathcal{H}_{B(n)}, \not{D}_{B(n)}), (C(SG), \mathcal{H}_\infty, \not{D}_\infty)) = 0.$$

Spectral Propinquity Metric Convergence of Spectral Triples on a Piecewise C^1-fractal Curve (L., Lapidus, Latrémolière)

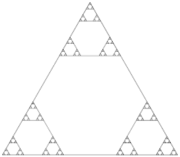
Let $\{R_j\}_{j\in\mathbb{N}}$ be a parameterization of X as a piecewise C^1-fractal curve and $B(n)$ an approximation sequence of X compatible with $\{R_j\}_{j\in\mathbb{N}}$. Denote the Lapidus and Sarhad spectral triple on X, $(C(X), \bigoplus_{j\geq 1} H_{l_j}, \bigoplus_{j\geq 1} D_{l_j})$, by $(C(SG), H_\infty, D_\infty)$ and $(C(X_{B(n)}), \bigoplus_{j=1}^{B(n)} H_{l_j}, \bigoplus_{j=1}^{B(n)} D_{l_j})$, by $(C(X_{B(n)}), H_{B(n)}, D_{B(n)})$.

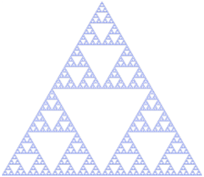
When equipped with $L_{D_\infty}(a) := \|[D_\infty, \pi(a)]\|_{B(H_\infty)}$, $(C(X), L_{D_\infty})$ is a quantum compact metric space. Similarly, $(C(X_{B(n)}), L_{D_{B(n)}})$ is also a quantum compact metric space.

Moreover,

$$\lim_{n\to\infty} \Lambda^{spec}((C(X_{B(n)}), H_{B(n)}, D_{B(n)}), (C(X), H_\infty, D_\infty)) = 0.$$

Work in Progress: The Stretched Sierpinski Gasket of Parameter α, $0 < \alpha < \frac{1}{3}$





#0 !

SG $\overset{Haus_{d_{eucl}}}{!}$ SG

$(C(SG\ ); H_{SG_\alpha}; Ð_{SSG_\alpha})$ $\overset{\text{Spectral Propinquity}}{!}$ $(C(SG); H_{SG}; Ð_{SG})$

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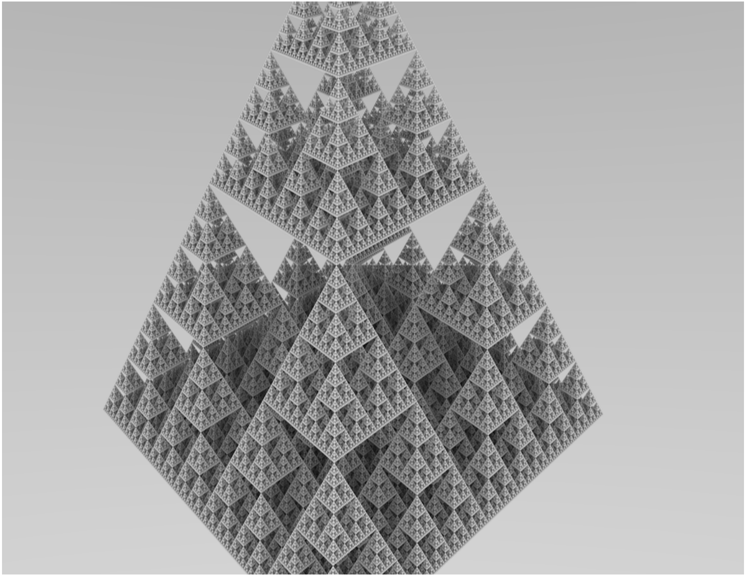
Future Work



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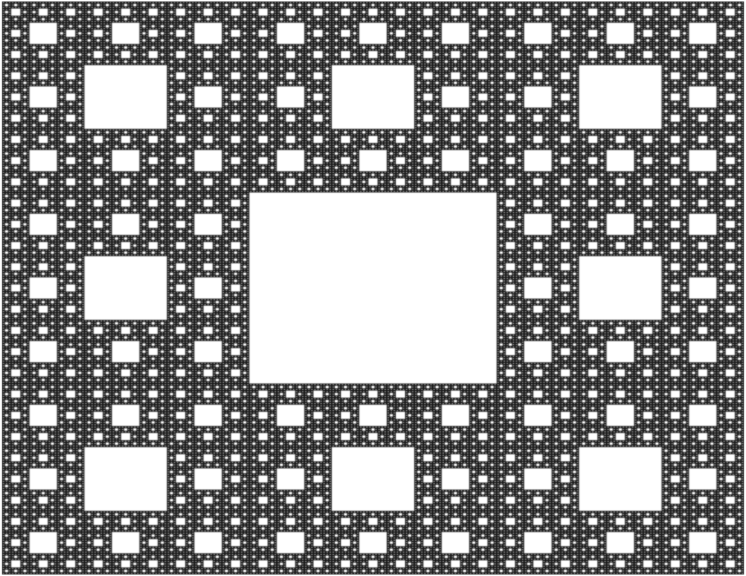

Future Work

Definition and Study of a "Fractal Manifold"

Classification of C -algebras on Fractals

Approximation of Laplacians on Fractals

Noncommutative Fractality

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