Limit theorems for multiplicative convolutions in classical and bi-free probabilities

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This talk summarizes limit theorems in

- T. Hasebe and H.-W. Huang, Limit theorems and wrapping transforms in bi-free probability theory, arXiv:2007.02775

Limit theorems for sums of indep. r.v.s

Motivation: Recall that CLT says, for iid X_k with mean m and variance $\sigma^2 > 0$,

$$\text{The law of } \frac{X_1}{\sqrt{n\sigma^2}} + \cdots + \frac{X_n}{\sqrt{n\sigma^2}} - \frac{\sqrt{n}m}{\sigma} \to N(0,1).$$

"Sum of many small indep. random variables" + "a shift (can be large)" converges to a non-zero limit.

Generalization:

$$X_{n,1} + X_{n,2} + \cdots + X_{n,k_n} + a_n,$$

where "$X_{n,k} \simeq 0$ uniformly on k" when n is large. Assume that $k_n \to \infty$ and $X_{n,1}, \ldots, X_{n,k_n}$ are independent, but not necessarily identically distributed. In terms of probability measures,

$$\mu_{n,1} * \cdots * \mu_{n,k_n} * \delta_{a_n}$$

where $\mu_{n,k} \simeq \delta_0$ (uniformly on k). What is the possible limit distributions?

Limit theorems and ID distributions

Theorem (Lévy–Khintchine representation)

Let $\mu \in \mathcal{P}(\mathbb{R})$. The following are equivalent.

(1) $\mu \in \mathcal{ID}(*)$ ($\overset{\text{def}}{\Longleftrightarrow} \forall n \in \mathbb{N}, \exists \mu_n, \mu = \mu_n * \mu_n * \cdots * \mu_n$.)

(2) *The Fourier transform of μ has the form*

$$\widehat{\mu}(z) = \exp\left(\mathrm{i}\gamma z + \int_{\mathbb{R}} \left(e^{\mathrm{i}zx} - 1 - \frac{\mathrm{i}zx}{1+x^2}\right) \frac{1+x^2}{x^2}\,\sigma(\mathrm{d}x)\right),$$

where $\gamma \in \mathbb{R}$ and σ is a finite measure on $\mathbb{R}$.

- The $*$-ID distribution μ with pair (γ, σ) is denoted by $\mu_*^{(\gamma,\sigma)}$.

Theorem (Bawly '36 and Khintchine '37)

If $\mu_{n1} * \cdots * \mu_{n,k_n} * \delta_{a_n} \to \mu$ weakly then $\mu \in \mathcal{ID}(*)$. Conversely, any ID distribution can be obtained in this way for some $\{a_n\}$ and $\{\mu_{n,k}\}$.

General limit theorems for free convolution

The concept of ID can be defined for other convolutions.

- (Bercovici-Voiculescu 93) μ is $\boxplus$-infinitely divisible

$$\overset{\text{def}}{\Longleftrightarrow} \forall n \in \mathbb{N}, \exists \mu_n \text{ such that } \mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ fold free convolution}}.$$

- μ is $\boxplus$-infinitely divisible iff

$$\varphi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+zx}{z-x} \sigma(dx) \qquad (\varphi_\mu : \text{the Voiculescu transform})$$

for some $\gamma \in \mathbb{R}$ and a finite nonnegative measure σ. The same parameter (γ, σ) appears both in classical and free probabilities.

Theorem (Bercovici & Pata '00)

μ_{nk}: as before, $a_n \in \mathbb{R}$. If $\mu_{n,1} \boxplus \cdots \boxplus \mu_{n,k_n} \boxplus \delta_{a_n} \to \mu$ weakly then $\mu \in \mathcal{ID}(\boxplus)$. Conversely, any $\boxplus$-infinitely divisible distribution can be obtained in this way for some $\{a_n\}$ and $\{\mu_{n,k}\}$.

Equivalence between limit theorems

Theorem (Gnedenko ∼'40, Chistyakov & Goetze08)

$\{\mu_{n,k}\}_{n,k}$: *as before (if n is large then $\mu_{n,k}$ is close to δ_0 uniformly on k),* $a_n, \gamma \in \mathbb{R}$, σ: *finite measure. The following statements are equivalent:*

- $\mu_{n,1} * \cdots * \mu_{n,k_n} * \delta_{a_n} \to \mu_*^{(\gamma,\sigma)}$;
- $\mu_{n,1} \boxplus \cdots \boxplus \mu_{n,k_n} \boxplus \delta_{a_n} \to \mu_{\boxplus}^{(\gamma,\sigma)}$;
- *(Gnedenko's condition) for a fixed* $r > 0$, $a_{n,k} = \int_{|x|<r} x \, d\mu_{n,k}(x)$ *and* $\mathring{\mu}_{n,k} := \delta_{-a_{n,k}} * \mu_{n,k}$,

$$\sum_{k=1}^{k_n} \frac{x^2 \, d\mathring{\mu}_{n,k}(x)}{1+x^2} \xrightarrow{\mathrm{w}} \sigma, \quad a_n + \sum_{k=1}^{k_n} \left(a_{n,k} + \int_{\mathbb{R}} \frac{x}{1+x^2} \, \mathring{\mu}_{n,k}(\mathrm{d}x) \right) \to \gamma,$$

- A similar equivalence holds for boolean convolution (Wang 08), but does not hold for monotone convolution (Franz & H. & Schleissinger).

Bi-free independence (Voiculescu '14)

$(H_i, \xi_i), i \in I$: pointed Hilbert spaces, $H_i = \mathbb{C}\xi_i \oplus H_i^\circ$

Let

$$(H, \xi) = *_{i \in I}(H_i, \xi_i),$$

where

$$H = \mathbb{C}\xi \oplus \bigoplus_{i_1 \neq \cdots \neq i_n, n \in \mathbb{N}} H_{i_1}^\circ \otimes \cdots \otimes H_{i_n}^\circ \quad \text{(Free product Hilbert space)}$$

For each i there is a canonical isomorphism

$$V_i : H \simeq H_i \otimes H(\ell, i),$$

and define a homomorphism $\lambda_i : B(H_i) \to B(H)$ via

$$\lambda_i(S) := V_i^*(S \otimes I)V_i.$$

The family $\{\lambda_i(B(H_i))\}_i$ is free in $(B(H), \xi)$.

Bi-free independence

$$H = \mathbb{C}\xi \oplus \bigoplus_{i_1 \neq \cdots \neq i_n, n \in \mathbb{N}} H_{i_1}^{\circ} \otimes \cdots \otimes H_{i_n}^{\circ} \quad \text{(Free product Hilbert space)}$$

For each i there is another canonical isomorphism (from the right)

$$W_i : H \simeq H(r, i) \otimes H_i,$$

and define a homomorphism $\rho_i : B(H_i) \to B(H)$ via

$$\rho_i(T) := W_i^*(I \otimes T)W_i.$$

The family $\{\rho_i(B(H_i))\}_i$ is also free in $(B(H), \xi)$. For $S_i, T_i \in B(H_i)$, the computation formula for the mixed moments of $\{(\lambda_i(S_i), \rho_i(T_i))\}_i$ is called bi-freeness.

Bi-free convolutions

Assume (a_1, b_1) and (a_2, b_2) are bi-free, $[a_i, b_j] = 0$ (bi-partite)

- Case 1: self-adjoint

  If (a_i, b_i) has a probab measure μ_i on $\mathbb{R}^2$ then

  the probab measure of $(a_1 + a_2, b_1 + b_2)$ is denoted by $\mu_1 \boxplus\boxplus \mu_2$
- Case 2: unitary

  If (a_i, b_i) has a probab measure μ_i on $\mathbb{T}^2$ then

  the probab measure of $(a_1 a_2, b_1 b_2)$ is denoted by $\mu_1 \boxtimes\boxtimes \mu_2$
- Infinite divisibility is defined similarly.
- The bi-free Voiculescu transform ϕ_μ linearizes $\boxplus\boxplus$, and the bi-free partial S-transform S_μ satisfies $S_{\mu_1 \boxtimes\boxtimes \mu_2}(z, w) = S_{\mu_1}(z, w) S_{\mu_2}(z, w)$. The bi-free partial Σ-transforms is defined by

$$\Sigma_\mu(z, w) = S_\mu\left(\frac{z}{1-z}, \frac{w}{1-w}\right)$$

2-dimensional LK repr

Let ϕ_μ be the bi-free Voiculescu transform.

Theorem (H.-Huang-Wang '18)

If $\mu \in \mathcal{ID}(\boxplus\boxplus)$ *then*

$$\phi_\mu(1/z, 1/w) = v_1 z + v_2 w + \left(a_{11} z^2 + a_{12} z w + a_{22} w^2\right) + \int_{\mathbb{R}^2} \left[\frac{1}{(1 - x_1 z)(1 - x_2 w)} - 1 - \frac{x_1 z + x_2 w}{1 + \|\boldsymbol{x}\|^2}\right] d\tau(\boldsymbol{x}),$$

where $(v_1, v_2) \in \mathbb{R}^2$, $A = (a_{ij})$ *is a positive semi-definite matrix,* τ *is a Lévy measure on* $\mathbb{R}^2 \setminus \{0\}$, *namely* $\int 1 \wedge \|\boldsymbol{x}\|^2 d\tau < \infty$.

This is the complete analogy of classical LK rep.

$$\widehat{\mu}(\mathbf{u}) = \exp\left[i\langle \mathbf{u}, \mathbf{v}\rangle - \frac{1}{2}\langle \mathbf{A}\mathbf{u}, \mathbf{u}\rangle + \int_{\mathbb{R}^2} \left(e^{i\langle \mathbf{u}, \mathbf{x}\rangle} - 1 - \frac{i\langle \mathbf{u}, \mathbf{x}\rangle}{1 + \|\mathbf{x}\|^2}\right) d\tau(\mathbf{x})\right]$$

Cf: Huang-Wang '16, Gu-Huang-Mingo '16 for earlier works

Theorem (H-Huang-Wang '18)

Let $\{\mu_{nk}\}$ *be infinitesimal array on* $\mathbb{R}^2$ *and* $\{\boldsymbol{a}_n\} \subset \mathbb{R}^2$ *If* $\mu_{n1} \boxplus\!\!\boxplus \cdots \boxplus\!\!\boxplus \mu_{n,k_n} \boxplus\!\!\boxplus \delta_{\boldsymbol{a}_n}$ *converges weakly, then the limit is ID.*

Theorem (H-Huang-Wang '18)

$\{\mu_{nk}\}$*: infinitesimal array on* $\mathbb{R}^2$*;* $\{\boldsymbol{v}_n\} \subset \mathbb{R}^2$*;* $(\boldsymbol{v}, \mathbf{A}, \tau)$ *given. TFAE.*

1. $\mu_{n1} * \cdots * \mu_{n,k_n} * \delta_{\boldsymbol{v}_n} \xrightarrow{\text{w}} \mu_*^{(\boldsymbol{v},\mathbf{A},\tau)}$.
2. $\mu_{n1} \boxplus\!\!\boxplus \cdots \boxplus\!\!\boxplus \mu_{n,k_n} \boxplus\!\!\boxplus \delta_{\boldsymbol{v}_n} \xrightarrow{\text{w}} \mu_{\boxplus\!\boxplus}^{(\boldsymbol{v},\mathbf{A},\tau)}$.
3. *A Gnedenko type condition (due to Rvaceva)*

Huang-Wang '16 proved an equivalence of (2) and (3) in a quite different form and assuming iid (μ_{nk} is independent of k)

For the equivalence of (1) and (3), see Rvaceva '62 or

- M.M. Meerschaert and H.-P. Scheffler, *Limit distributions for sums of independent random vectors*. John Wiley & Sons, Inc., 2001.

Unit circle case ($\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$)

Suppose $X_{n,1}, \ldots, X_{n,k_n}$ are $\mathbb{T}$-valued, indep. and $k_n \to \infty$. We think of

$$\gamma_n X_{n,1} X_{n,2} \cdots X_{n,k_n},$$

where $\gamma_n \in \mathbb{T}$, and for every $\epsilon > 0$

$$\lim_{n\to\infty} \sup_{1\le k\le k_n} \mathbb{P}[|X_{n,k} - 1| \ge \epsilon] = 0 \quad (\text{infinitesimal array})$$

In terms of probability measures, multiplicative convolution $\mu \circledast \nu$ is characterized by

$$\int_{\mathbb{T}} f(x) d(\mu \circledast \nu)(x) = \int_{\mathbb{T}^2} f(xy) d\mu(x) d\nu(y)$$

Our problem is, for $\mu_{n,k}$ the law of X_{nk}, when

$$\mu_{n,1} \circledast \cdots \circledast \mu_{n,k_n} \circledast \delta_{\gamma_n}$$

converges weakly.

Lévy-Khintchine representation

A prob. meas. μ on $\mathbb{T}$ is ID if $\forall n, \exists \mu_n$ such that $\mu = \mu_n^{\circledast n}$.

Theorem (Ref: Parthasarathy '67)

$\{\mu_{nk}\}$: infinitesimal array; $\gamma_n \in \mathbb{T}$. If $\mu_{n,1} \circledast \cdots \circledast \mu_{n,k_n} \circledast \delta_{\gamma_n} \to \mu$ weakly then μ is ID. Conversely, any ID distribution on $\mathbb{T}$ can be obtained in this way for some $\{\mu_{n,k}\}$ and $\{\gamma_n\}$.

Theorem (Lévy-Khintchine representation. Ref: Parthasarathy '67)

If $\int_{\mathbb{T}} x \, d\mu(x) \neq 0$ and $\mu \in \mathcal{ID}(\circledast)$ then

$$\int_{\mathbb{T}} x^n \, d\mu(x) = \gamma^n \exp\left[\int_{\mathbb{T}} \frac{x^n - 1 - in\Im(x)}{1 - \Re(x)} d\sigma(x)\right]$$

for some $\gamma \in \mathbb{T}$, a finite measure σ on $\mathbb{T}$. Write $\mu = \mu_{\circledast}^{(\gamma,\sigma)}$.

Non-uniqueness of the Lévy measure

Theorem (Lévy-Khintchine representation. Ref: Parthasarathy '67)

If $\int_{\mathbb{T}} x \, d\mu(x) \neq 0$ *and* μ *is ID then*

$$\int_{\mathbb{T}} x^n \, d\mu(x) = \gamma^n \exp\left[\int_{\mathbb{T}} \frac{x^n - 1 - in\Im(x)}{1 - \Re(x)} d\sigma(x)\right]$$

for some $\gamma \in \mathbb{T}$*, a finite measure* σ *on* $\mathbb{T} \setminus \{1\}$*. Write* $\mu = \mu_{\circledast}^{(\gamma,\sigma)}$*.*

Remark: γ is unique, but σ is not unique in general!

Example (Cébron '16): $\mu_{\circledast}^{(1,\pi\delta_i)} = \mu_{\circledast}^{(1,\pi\delta_{-i})}$, because

$$\int_{\mathbb{T}} x^n \, d\mu_{\circledast}^{(1,\pi\delta_i)}(x) = \int_{\mathbb{T}} x^n \, d\mu_{\circledast}^{(1,\pi\delta_{-i})}(x), \quad \forall n \in \mathbb{Z}$$

because

$$\pi(i^n - 1 - in) - \pi((-i)^n - 1 + in) \equiv 0 \mod 2\pi i.$$

Comparing classical and free

Theorem (Chistyakov & Goetze '08)

Given $r \in (0,1), \mu_{n,k}, \xi_n, (\gamma,\sigma)$*, the following statements are equivalent:*

1. $\mu_{n1} \boxtimes \cdots \boxtimes \mu_{n,k_n} \boxtimes \delta_{\xi_n} \to \mu_{\boxtimes}^{(\gamma,\sigma)}$;
2. *for* $a_{nk} = \int_{|\arg x|<r} (\arg x)\, d\mu_{nk}(x)$ *and* $\mathring{\mu}_{nk} := \delta_{e^{-ia_{nk}}} \circledast \mu_{nk}$,
   - $\xi_n \exp(i \sum_{k=1}^{k_n} \left(a_{nk} + \int_{\mathbb{T}} \Im(x)\, \mathring{\mu}_{nk}(\mathrm{d}x)\right) \to \gamma$,
   - $\sigma_n := \sum_{k=1}^{k_n} (1 - \Re(x)) d\mathring{\mu}_{nk}(x) \xrightarrow{\mathrm{w}} \sigma$.

Let $\mathcal{L}(\sigma) := \{\sigma'$ finite measure on $\mathbb{T} : \mu_{\circledast}^{(1,\sigma')} = \mu_{\circledast}^{(1,\sigma)}\}$.

Theorem (H.-Huang, can be extended to $\mathbb{T}^d$.)

Notations as above. The following statements are equivalent:

1. $\mu_{n1} \circledast \cdots \circledast \mu_{n,k_n} \circledast \delta_{\xi_n} \to \mu_{\circledast}^{(\gamma,\sigma)}$;
2.
   - $\xi_n \exp(i \sum_{k=1}^{k_n} \left(a_{nk} + \int_{\mathbb{T}} \Im(x)\, \mathring{\mu}_{nk}(\mathrm{d}x)\right) \to \gamma$,
   - *the finite measures* σ_n *form a relatively compact family and any limit point of* $\{\sigma_n\}$ *is contained in* $\mathcal{L}(\sigma) = \{\sigma' : \mu_{\circledast}^{(1,\sigma')} = \mu_{\circledast}^{(1,\sigma)}\}$.

Recall that $\mathcal{L}(\sigma) = \{\sigma'$ finite measure on $\mathbb{T} : \mu_{\circledast}^{(1,\sigma')} = \mu_{\circledast}^{(1,\sigma)}\}$.

Corollary (Chistyakov & Goetze '08 + Wang '08)

Suppose that $\mathcal{L}(\sigma) = \{\sigma\}$. The following statements are equivalent:

1. $\mu_{n1} \circledast \cdots \circledast \mu_{n,k_n} \circledast \delta_{\xi_n} \to \mu_{\circledast}^{(\gamma,\sigma)}$;
2. $\mu_{n1} \boxtimes \cdots \boxtimes \mu_{n,k_n} \boxtimes \delta_{\xi_n} \to \mu_{\boxtimes}^{(\gamma,\sigma)}$.

- 2 $\Rightarrow$ 1 holds without assuming $\mathcal{L}(\sigma) = \{\sigma\}$ (Chistyakov & Goetze).
- What kind of measure σ is unique? (next slides)

Uniqueness of Lévy measures

We say σ is unique if

$$\mathcal{L}(\sigma) = \{\sigma' \text{ finite measure on } \mathbb{T} : \mu_{\circledast}^{(1,\sigma')} = \mu_{\circledast}^{(1,\sigma)}\} = \{\sigma\}.$$

Wang '08: the normal distribution on $\mathbb{T}$ has a unique σ (=0)

Theorem (H.-Huang)

Let $\varphi_k \in (0,\pi)$ and let $c_k, d_k \geq 0$ for $k = 1, 2, \ldots, m$, and set $\alpha_k = e^{i\varphi_k}$. If $\cos\varphi_1, \ldots, \cos\varphi_m$ are algebraically independent over $\mathbb{Q}$, then the Lévy measure $\sum_{k=1}^{m}(c_k\delta_{\alpha_k} + d_k\delta_{\bar{\alpha}_k})$ is unique.

Corollary: the set of ID distributions with unique σ is weakly dense in ID.

Theorem (H.-Huang)

Let $\varphi_k \in (0,\pi)$ and let $c_k, d_k \geq 0$ for $k = 1, 2, \ldots, m$, and set $\alpha_k = e^{i\varphi_k}$. If $\cos\varphi_1, \ldots, \cos\varphi_m$ are algebraically independent over $\mathbb{Q}$, then $\sigma = \sum_{k=1}^{m}(c_k\delta_{\alpha_k} + d_k\delta_{\bar{\alpha}_k})$ is unique.

Outline: If ν' is another Lévy measure, the basic relation is

$$\int_{\mathbb{T}}(x^n - 1 - in\Im(x))d\nu(x) = \int_{\mathbb{T}}(x^n - 1 - in\Im(x))d\nu'(x) \mod 2\pi i, \quad \forall n \in \mathbb{Z}$$

(1) Prove that $\operatorname{supp}(\nu') \subseteq \{\alpha_k, \overline{\alpha_k} : k = 1, 2, \ldots, m\}$.

(2) Write $\nu' = \sum_{k=1}^{m}(c'_k\delta_{\alpha_k} + d'_k\delta_{\bar{\alpha}_k})$, and put it into the basic relation to obtain a system of linear equations for c'_k, d'_k (with unknown numbers from $\mod 2\pi i$). Prove that $c'_k + d'_k = c_k + d_k$.

(3) Construct a polynomial having the roots $\cos\varphi_k,\ k = 1, 2, \ldots, m$.

(4) Because the polynomial must be zero, we obtain $c'_k = c_k, d'_k = d_k$.

Idempotent distributions for ⊠⊠

In the theory of ID distributions on a locally compact abelian group, idempotent distributions ($\mu \circ \mu = \mu$) are important because they are obstruction for a Lévy-Khintchine repr. They are classified as

$$\{\text{idem. distributions}\} = \{\text{the normalized Haar measures on cpt subgrps}\}\ .$$

For free case, there are only two ⊠-idempotents δ_1 and m (the normalized Haar measure on $\mathbb{T}$).
We say that μ is ⊠⊠-idempotent if $\mu ⊠⊠ \mu = \mu$. Let

$$P(B) = \mathrm{m}\big(\{s \in \mathbb{T} : (s, \bar{s}) \in B\}\big), \qquad B \in \mathscr{B}_{\mathbb{T}^2},$$

Proposition

A ⊠⊠-idempotent distribution on $\mathbb{T}^2$ is one of five types $\delta_{(1,1)}$ (trivial one), $\mathrm{m} \times \delta_1$, $\delta_1 \times \mathrm{m}$, $\mathrm{m} \times \mathrm{m}$, *and* P.

Proposition

A $\boxtimes\boxtimes$-idempotent distribution in $\mathscr{P}_{\mathbb{T}^2}$ is one of five types $\delta_{(1,1)}$ (trivial one), $\mathrm{m} \times \delta_1$, $\delta_1 \times \mathrm{m}$, $\mathrm{m} \times \mathrm{m}$, *and* P.

Remark

Katsimpas introduced a bi-Haar unitary element, which has the distribution P^* defined by

$$P^*(B) = \mathrm{m}\big(\{s \in \mathbb{T} : (s,s) \in B\}\big)$$

Idea of the proof: use the operator model for bi-freeness and carefully look at the formula

$$m_{p,q}(\mu \boxtimes\boxtimes \mu) = m_{p,q}(\mu), \qquad (p,q) \in \mathbb{Z}^2,$$

where $m_{p,q}(\mu) = \int_{\mathbb{T}^2} s^p t^q \, d\mu(s,t)$.

Classification of $\boxtimes\boxtimes$-idempotents

We say that μ has a non-trivial $\boxtimes\boxtimes$-idempotent factor if $\mu = \lambda \boxtimes\boxtimes \nu$, where λ is a non-trivial $\boxtimes\boxtimes$-idempotent distr.

Theorem (Complete analogy with classical case)

In order that a measure $\nu \in \mathcal{ID}(\boxtimes\boxtimes)$ contains no non-trivial $\boxtimes\boxtimes$-idempotent factors, it is necessary and sufficient that $m_{1,0}(\nu) \neq 0 \neq m_{0,1}(\nu)$.

The set of such distributions is denoted by $\mathcal{ID}_\times(\boxtimes\boxtimes)$.

Corollary

Any measure $\nu \in \mathcal{ID}(\boxtimes\boxtimes) \setminus \mathcal{ID}_\times(\boxtimes\boxtimes)$ is either $\nu^{(1)} \times \mathrm{m}$, $\mathrm{m} \times \nu^{(2)}$, $\mathrm{m} \times \mathrm{m}$ or $P \circledast (\kappa_c \times \delta_1) = P \boxtimes\boxtimes (\kappa_c \times \delta_1)$, where $\nu^{(1)}$ and $\nu^{(2)}$ are in $\mathcal{ID}_\times(\boxtimes)$ and $c \in (\mathbb{D} \cup \mathbb{T}) \setminus \{0\}$, where $d\kappa_c(s) = \frac{1-|c|^2}{|1-\bar{c}s|^2}\,\mathrm{dm}(s)$.

⊠⊠-idempotents and LK repr

For $\nu \in \mathcal{ID}_\times(⊠⊠)$, let $\nu^{(1)}$ be the first marginal and $\nu^{(2)}$ the second. We use the exponential form of Σ-transform for free and bi-free:

$$\Sigma_{\nu^{(j)}}(\xi) = \exp[u_j(\xi)] \quad \text{and} \quad \Sigma_\nu(z,w) = \exp[u(z,w)],$$

In addition to the marginal formulas

$$u_j(\xi) = -i\arg\gamma_j + \int_{\mathbb{T}} \frac{1+\xi s}{1-\xi s}\,d\tilde{\lambda}_j(s), \qquad \xi \in \mathbb{D},$$

for $j = 1, 2$, Huang-Wang '18 obtained the following repr:

$$\begin{aligned}\frac{(1-z)(1-w)}{1-zw}u(z,w) &= \int_{\mathbb{T}^2} \frac{1+zs_1}{1-zs_1}\frac{1+ws_2}{1-ws_2}(1-\Re s_2)\,d\lambda_1(\boldsymbol{s}) \\ &\quad - i\int_{\mathbb{T}^2}\frac{1+zs_1}{1-zs_1}(\Im s_2)\,d\lambda_1(\boldsymbol{s}) - i\int_{\mathbb{T}^2}\frac{1+ws_2}{1-ws_2}(\Im s_1)\,d\lambda_2(\boldsymbol{s}) - L.\end{aligned}$$

We reformulate this in terms of a triplet.

We use the multi-index notation as $\boldsymbol{s}^{\boldsymbol{p}} = s_1^{p_1} s_2^{p_2}$, $\arg \boldsymbol{s} = (\arg s_1, \arg s_2)$.

Proposition (H-Huang)

For $\nu \in \mathcal{ID}_\times(\boxtimes\boxtimes)$ *write* $\Sigma_{\nu^{(j)}}(\xi) = \exp[u_j(\xi)], \Sigma_\nu(z,w) = \exp[u(z,w)]$. *Then there exist* $\boldsymbol{\gamma} \in \mathbb{T}^2$, *positive semidefinite* $\mathbf{A} \in M_2(\mathbb{R})$ *and a measure* ρ *such that* $1 - \Re(s_1), 1 - \Re(s_2) \in L^1(\rho)$ *(Lévy measure) and*

$$\frac{zwu(z,w)}{(1-zw)(1-z)(1-w)} - \frac{zu_1(z)}{(1-z)^2(1-w)} - \frac{wu_2(w)}{(1-z)(1-w)^2}$$
$$= \sum_{\boldsymbol{p} \in \mathbb{Z}^2_{\geq 0}} \left[i\langle \boldsymbol{p}, \arg \boldsymbol{\gamma} \rangle - \frac{1}{2}\langle \mathbf{A}\boldsymbol{p}, \boldsymbol{p} \rangle + \int_{\mathbb{T}^2} \left(\boldsymbol{s}^{\boldsymbol{p}} - 1 - i\langle \boldsymbol{p}, \Im \boldsymbol{s} \rangle \right) d\rho(\boldsymbol{s}) \right] z^{p_1} w^{p_2}$$

Write $\nu = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}, \mathbf{A}, \rho)}$.

To be compared with the classical formula (in $\mathbb{T}^d$): for $\nu \in \mathcal{ID}_\times(\circledast)$

$$\widehat{\nu}(\boldsymbol{p}) = \boldsymbol{\gamma}^{\boldsymbol{p}} \exp\left(-\frac{1}{2}\langle \mathbf{A}\boldsymbol{p}, \boldsymbol{p} \rangle + \int_{\mathbb{T}^d} \left(\boldsymbol{s}^{\boldsymbol{p}} - 1 - i\langle \boldsymbol{p}, \Im \boldsymbol{s} \rangle \right) d\rho(\boldsymbol{s}) \right), \quad \boldsymbol{p} \in \mathbb{Z}^d.$$

Gnedenko-type condition for ⊠⊠

Theorem (H-Huang)

Let $\mathscr{U}_\epsilon = \{\boldsymbol{s} \in \mathbb{T}^2 : \|\arg \boldsymbol{s}\| < \epsilon\}$*. For an infinitesimal array* $\{\nu_{nk}\} \subset \mathscr{P}_{\mathbb{T}^2}^{\times}$ *and a sequence* $\{\boldsymbol{\xi}_n\} \subset \mathbb{T}^2$*, define*

$$\boldsymbol{b}_{nk} = \exp\left[i \int_{\mathscr{U}_{1/2}} (\arg \boldsymbol{s})\, d\nu_{nk}(\boldsymbol{s})\right], \qquad d\mathring{\nu}_{nk}(\boldsymbol{s}) = d\nu_{nk}(\boldsymbol{b}_{nk}\boldsymbol{s}),$$

$$\boldsymbol{\gamma}_n = \boldsymbol{\xi}_n \exp\left[i \sum_{k=1}^{k_n}\left(\arg \boldsymbol{b}_{nk} + \int_{\mathbb{T}^2} (\Im \boldsymbol{s})\, d\mathring{\nu}_{nk}(\boldsymbol{s})\right)\right].$$

Given a triplet $(\boldsymbol{\gamma}, \mathbf{A}, \rho)$ *the following are equivalent.*

(1) $\delta_{\boldsymbol{\xi}_n}$ ⊠⊠ ν_{n1} ⊠⊠ $\cdots$ ⊠⊠ ν_{nk} *converges weakly to* $\nu_{⊠⊠}^{(\boldsymbol{\gamma},\mathbf{A},\rho)}$.

(2) $\rho_n = \sum_{k=1}^{k_n} \mathring{\nu}_{nk} \to \rho$, $\boldsymbol{\gamma}_n \to \boldsymbol{\gamma}$ *and for every* $\boldsymbol{p} \in \mathbb{Z}^2$

$$\langle \boldsymbol{p}, \mathbf{A}\boldsymbol{p}\rangle = \lim_{\epsilon\to 0} \overline{\lim_{n\to\infty}} \int_{\mathscr{U}_\epsilon} \langle \boldsymbol{p}, \Im \boldsymbol{s}\rangle^2 d\rho_n(\boldsymbol{s}) = \lim_{\epsilon\to 0} \underline{\lim_{n\to\infty}} \int_{\mathscr{U}_\epsilon} \langle \boldsymbol{p}, \Im \boldsymbol{s}\rangle^2 d\rho_n(\boldsymbol{s}).$$

Combining the limit theorems for bi-free and classical convolutions we obtain:

Corollary (H-Huang)

In the setting above, consider the two statements

(1) $\delta_{\boldsymbol{\xi}_n} \circledast \nu_{n1} \circledast \cdots \circledast \nu_{nk}$ *converges weakly to* $\nu_{\circledast}^{(\boldsymbol{\gamma},\mathbf{A},\rho)}$

(2) $\delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes \nu_{n1} \boxtimes\boxtimes \cdots \boxtimes\boxtimes \nu_{nk}$ *converges weakly to* $\nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},\mathbf{A},\rho)}$.

Then (2) implies (1). The converse is also true if ρ *is the unique Lévy measure for* $\nu_{\circledast}^{(\boldsymbol{\gamma},\mathbf{A},\rho)}$.

In conclusion:

- For additive convolutions, the transfer principle is perfect in both 1- and 2-dim.
- For multiplicative convolutions, the non-uniqueness of the Lévy measure is the obstruction for the transfer principle from classical to free/bi-free.

Wrapping transform

The following result is a 2-dim version of Cébron's theorem (2016). Let

$$W\colon \mathbb{R}^2 \to \mathbb{T}^2, \qquad W(x_1, x_2) = (e^{ix_1}, e^{ix_2}).$$

Theorem (H-Huang)

Let $(\boldsymbol{v}, \mathbf{A}, \tau)$ be a Lévy triplet, and let $\{\mu_{nk}\}$ be an infinitesimal triangular array on $\mathbb{R}^2$ and $\{\boldsymbol{v}_n\}$ a sequence of vectors in $\mathbb{R}^2$. If

$$\mu_{n1} \boxplus\boxplus \cdots \boxplus\boxplus \mu_{n,k_n} \boxplus\boxplus \delta_{\boldsymbol{v}_n} \xrightarrow{\mathrm{w}} \mu_{\boxplus\boxplus}^{(\boldsymbol{v},\mathbf{A},\tau)},$$

then

$$\nu_{n1} \boxtimes\boxtimes \cdots \boxtimes\boxtimes \nu_{n,k_n} \boxtimes\boxtimes \delta_{\boldsymbol{\xi}_n} \xrightarrow{\mathrm{w}} \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},\mathbf{A},\rho)},$$

where $\nu_{nk} = \mu_{nk} W^{-1}, \boldsymbol{\xi}_n = W(\boldsymbol{v}_n)$, and

$$\boldsymbol{\gamma} = W\left(\boldsymbol{v} + \int_{\mathbb{R}^2} \left(\sin(\boldsymbol{x}) - \frac{\boldsymbol{x}}{1 + \|\boldsymbol{x}\|^2}\right) d\tau(\boldsymbol{x})\right), \qquad \rho = 1_{\mathbb{T}^2 \setminus \{\mathbf{1}\}} \tau W^{-1}.$$

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