The Matsumoto-Yor Property in free probability

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The Matsumoto-Yor property in classic probability

Let X, Y be positive independent random variables and let

$$U = \frac{1}{X+Y}, V = \frac{1}{X} - \frac{1}{X+Y}. \quad (1)$$

Matsumoto-Yor 2001 [6]

If X has the Generalized Inverse Gaussian law $GIG(-p, a, b)$, Y has the Gamma law $G(p, a)$, where $a, b, p > 0$, then U and V are independent and distributed $GIG(-p, b, a)$, $G(p, b)$ respectively.

We recall that

$$G(p, a) \sim \frac{a^p}{\Gamma(p)} x^{p-1} e^{-ax} \mathbb{1}_{(0,\infty)}(x), \text{ where } a, p > 0.$$

$$GIG(p, a, b) \sim \frac{(a/b)^{p/2}}{2K_p(2\sqrt{ab})} x^{p-1} e^{-ax-\frac{b}{x}} \mathbb{1}_{(0,\infty)}(x), \text{where } a, b > 0, p \in \mathbb{R}.$$

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Characterization theorem

Letac & Wesołowski 2000 [5]

Let X, Y be positive, independent and non-degenerate random variables and U, V be defined as in (1). If U and V are independent then $X \sim GIG(-p, a, b)$ and $Y \sim G(p, a)$ for some $a, b, p > 0$.

Notation

- $(\mathcal{A},\varphi)$- W^* probability space i.e. $\mathcal{A}$ is a finite von Neumann algebra and φ is a positive, tracial and faithful state such that $\varphi(\mathbb{I}) = 1$.
- If $\mathcal{B} \subseteq \mathcal{A}$ we denote $\varphi(\cdot \mid \mathcal{B}) : \mathcal{A} \to \mathcal{B}$ the conditional expectation with respect to $\mathcal{B}$.
- If $\mathbb{X} \in \mathcal{A}$ is self-adjoint with distribution $\mu_{\mathbb{X}}$ then for $z \in \mathbb{C}^+$

$$G_{\mathbb{X}}(z) = \varphi\left((z\mathbb{I} - \mathbb{X})^{-1}\right) = \int_{\mathbb{R}} \frac{\mu_{\mathbb{X}}(dx)}{z - x}$$

  is the Cauchy-Stieltjes transform of $\mathbb{X}$.
- $r_{\mathbb{X}}(z) = G_{\mathbb{X}}^{-1}(z) - \frac{1}{z}$ is the r-transform of $\mathbb{X}$.

The Matsumoto-Yor property in free probability characterizes free-Poisson (Marchenko Pastur) and free-GIG distributions.

The free-Poisson distribution $\nu = \nu(\lambda, \gamma)$ with $\lambda \geq 0, \gamma > 0$

$$\nu(\lambda, \gamma) = \max\{0, 1-\lambda\}\delta_0 + \lambda\nu_1,$$

where ν_1 is a measure with density

$$\frac{1}{2\pi\gamma x}\sqrt{4\lambda\gamma^2 - (x - \gamma(1+\lambda))^2}\ \mathbb{1}_{(\gamma(1-\sqrt{\lambda})^2, \gamma(1+\sqrt{\lambda})^2)}(x).$$

The r-transform of the free Poisson distribution $\nu(\lambda, \gamma)$ is equal

$$r_{\nu(\lambda,\gamma)}(z) = \frac{\lambda\gamma}{1-\gamma z}.$$

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Definition (Féral 2006 [2])

The free Generalized Inverse Gaussian distribution is a probability measure $\mu = \mu(\lambda, \alpha, \beta)$, with $\alpha, \beta > 0, \lambda \in \mathbb{R}$, which is compactly supported on the interval $[a, b]$ and has the density

$$\frac{d\mu}{dx} = \frac{1}{2\pi}\sqrt{(x-a)(b-x)}\left(\frac{\alpha}{x} + \frac{\beta}{\sqrt{ab}x^2}\right),$$

where (a, b) such that $0 < a < b$ is the unique solution of

$$\begin{cases} 1 - \lambda + \alpha\sqrt{ab} - \beta\frac{a+b}{ab} & = 0, \\ 1 + \lambda + \frac{\beta}{\sqrt{ab}} - \alpha\frac{a+b}{2} & = 0. \end{cases}$$

Szpojankowski 2017 [8]

Let $\mathbb{X}$ and $\mathbb{Y}$ be free self-adjoint, positive and non-degenerate random variables and let

$$\mathbb{U} = (\mathbb{X} + \mathbb{Y})^{-1} \text{ and } \mathbb{V} = \mathbb{X}^{-1} - (\mathbb{X} + \mathbb{Y})^{-1}.$$

Then $\mathbb{U}$ and $\mathbb{V}$ are free if and only if $\mathbb{X}$ has the free-GIG distribution $\mu(-\lambda, \alpha, \beta)$ and the distribution of $\mathbb{Y}$ is free-Poisson $\nu(\lambda, 1/\alpha)$ for some parameters $\alpha, \beta > 0$ and $\lambda \in \mathbb{R}$. In this case $\mathbb{U}$ and $\mathbb{V}$ have $\mu(-\lambda, \beta, \alpha)$ and $\nu(\lambda, 1/\beta)$ distributions respectively.

The main result (M.Ś. 2021 [7])

Let $\mathbb{X}$ and $\mathbb{Y}$ be free, positive, self-adjoint and non-degenerate random variables and let $\mathbb{U} = (\mathbb{X} + \mathbb{Y})^{-1}$ and $\mathbb{V} = \mathbb{X}^{-1} - (\mathbb{X} + \mathbb{Y})^{-1}$. If the following conditions are satisfied

$$\varphi\left(\mathbb{V}^{-1} \mid \mathbb{U}\right) = d\mathbb{I},\ \varphi\left(\mathbb{V}^{-2} \mid \mathbb{U}\right) = h\mathbb{I}$$

for some constants d and h, then $h > d^2$ and $\mathbb{X}$ has the free-GIG distribution $\mu\left(-\frac{h}{h-d^2}, \frac{\gamma d^2}{h-d^2}, \frac{d^3}{h-d^2}\right)$ and $\mathbb{Y}$ has the free Poisson distribution $\nu\left(\frac{h}{h-d^2}, \frac{h-d^2}{d^2\gamma}\right)$, where γ is some positive constant.

About the proof

Two main tools used in the proof are

- Subordination of free additive convolution.
- Boolean cumulants.

Let $\mathbb{X}, \mathbb{Y}$ be free. We denote $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$-the subordination functions i.e.

$$G_{\mathbb{X}+\mathbb{Y}}(z) = G_{\mathbb{X}}(\omega_1(z)) = G_{\mathbb{Y}}(\omega_2(z)).$$

Biane 1998 [1]

If $\mathbb{X}$ and $\mathbb{Y}$ are free, self-adjoint random variables, then for all $z \in \mathbb{C}^+$

$$\varphi\left((z\mathbb{I} - \mathbb{X} - \mathbb{Y})^{-1} \mid \mathbb{X}\right) = (\omega_1(z) - \mathbb{X})^{-1}.$$

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Definition

For $n \geq 1$ the Boolean cumulant functional $\beta_n : \mathcal{A}^n \to \mathbb{C}$ is defined recursively by

$$\forall \mathbb{X}_1, \ldots, \mathbb{X}_n \in \mathcal{A} : \varphi\left(\mathbb{X}_1 \cdot \ldots \cdot \mathbb{X}_n\right) = \sum_{\pi \in Int(n)} \beta_\pi\left(\mathbb{X}_1, \ldots, \mathbb{X}_n\right),$$

where $Int(n)$ is the set of interval partitions of $\{1, 2, \ldots, n\}$ and for $\pi = \{B_1, \ldots, B_k\}$

$$\beta_\pi\left(\mathbb{X}_1, \ldots, \mathbb{X}_n\right) = \prod_{j=1}^{k} \beta_{|B_j|}\left(\mathbb{X}_i : i \in B_j\right).$$

Sketch of the proof

Let us denote

- $\mathbb{T} = \mathbb{U}^{-1} = \mathbb{X} + \mathbb{Y}$. By positivity $\varphi\left(\cdot \mid \mathbb{T}\right) = \varphi\left(\cdot \mid \mathbb{U}\right)$.
- $\gamma = \varphi(\mathbb{Y}^{-1})$ and $\beta = \varphi(\mathbb{U})$.

Lemma 1

The condition

$$\varphi\left(\mathbb{V}^{-1} \mid \mathbb{U}\right) = d\mathbb{I},$$

implies that

$$\frac{1}{\omega_2(z)}\left(\gamma + G_{\mathbb{X}+\mathbb{Y}}(z)\right) = \frac{d\beta}{z^2} + \frac{\gamma}{z} + \left(\frac{d}{z^2} + \frac{1}{z}\right) G_{\mathbb{X}+\mathbb{Y}}(z)$$

for all $z \in \mathbb{C}^+$.

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for all $z \in \mathbb{C}^+$.

Lemma 2

The condition

$$\varphi\left(\mathbb{V}^{-2} \mid \mathbb{U}\right) = h\mathbb{I},$$

implies that

$$\varphi(\mathbb{X}^2)B(z) + A(z)^2\left(\omega_1^2(z)G_{\mathbb{X}+\mathbb{Y}}(z) - \omega_1(z) - \varphi(\mathbb{X})\right) = \\ = h\left(\frac{\varphi(\mathbb{U}^2)}{z} + \frac{\beta}{z^2} + \frac{1}{z^2}G_{\mathbb{X}+\mathbb{Y}}(z)\right)$$

for all $z \in \mathbb{C}^+$, where

$$A(z) = \frac{1}{\omega_2(z)} + \frac{\gamma}{\omega_2(z)G_{\mathbb{X}+\mathbb{Y}}(z)},$$

$$B(z) = \frac{\varphi(\mathbb{Y}^{-2}) - \gamma A(z)}{\omega_2(z)}.$$

The proof of Lemma 2 will require several formulas for Boolean cumulants.

Lehner, Szpojankowski 2019 [4]

Assume we are given two collections of random variables $\{\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n\}$ and $\{\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_n\}$ that are free, $n \geq 1$. then

$$\varphi\left(\mathbb{Y}_1\mathbb{X}_1 \ldots \mathbb{Y}_n\mathbb{X}_n\right) = \\ = \sum_{k=0}^{n-1} \sum_{0=j_0<j_1<\ldots<j_{k+1}=n} \varphi\left(\mathbb{X}_{j_1} \ldots \mathbb{X}_{j_{k+1}}\right) \cdot \\ \cdot \prod_{l=0}^{k} \beta_{2(j_{l+1}-j_l)-1}\left(\mathbb{Y}_{j_l+1}, \mathbb{X}_{j_l+1}, \ldots, \mathbb{X}_{j_{l+1}-1}, \mathbb{Y}_{j_{l+1}}\right)$$

Fevrier et al. 2020 [3]

Assume we are given two collections of random variables $\{\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_{n+1}\}$ and $\{\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_n\}$ that are free, $n \geq 1$. then

$$\beta_{2n+1}\left(\mathbb{X}_1, \mathbb{Y}_1, \ldots, \mathbb{X}_n, \mathbb{Y}_n, \mathbb{X}_{n+1}\right) = \\ = \sum_{k=2}^{n+1} \sum_{1=j_1<\ldots<j_k=n} \beta_k\left(\mathbb{X}_{j_1}, \ldots, \mathbb{X}_{j_k}\right) \cdot \\ \cdot \prod_{l=1}^{k-1} \beta_{2(j_{l+1}-j_l)-1}\left(\mathbb{Y}_{j_l}, \mathbb{X}_{j_l+1}, \mathbb{Y}_{j_l+1}, \ldots, \mathbb{X}_{j_{l+1}-1}, \mathbb{Y}_{j_{l+1}-1}\right).$$

Lehner. Szpojankowski [4]

$$\omega_1(z) = z - \sum_{n=0}^{\infty} \beta_{2n+1}\left(\mathbb{Y}, (z\mathbb{I} - \mathbb{X})^{-1}, \mathbb{Y}, \ldots, (z\mathbb{I} - \mathbb{X})^{-1}, \mathbb{Y}\right)$$

in some neighborhood of infinity in $\mathbb{C}^+$.

M.Ś. [7]

$$\frac{1}{\omega_2(z)} = \sum_{n=0}^{\infty} \beta_{2n+1}\left((z\mathbb{I} - \mathbb{X})^{-1}, \mathbb{Y}, \ldots, \mathbb{Y}, (z\mathbb{I} - \mathbb{X})^{-1}\right)$$

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Under assumptions of the main theorem $G_{\mathbb{X}+\mathbb{Y}}(z), \omega_1(z), \omega_2(z)$ satisfy equations

(1) $z = \omega_1(z) + \omega_2(z) - \frac{1}{G_{\mathbb{X}+\mathbb{Y}}(z)}$.

(2) $(\gamma + G_{\mathbb{X}+\mathbb{Y}}(z)) \left(\frac{1}{\omega_2(z)} - \frac{1}{z} \right) = \frac{d}{z^2} (\beta + G_{\mathbb{X}+\mathbb{Y}}(z))$.

(3) $\varphi(\mathbb{X}^2) B(z) + A(z)^2 \left(\omega_1^2(z) G_{\mathbb{X}+\mathbb{Y}}(z) - \omega_1(z) - \varphi(\mathbb{X}) \right) = h \left(\frac{\varphi(\mathbb{U}^2)}{z} + \frac{\beta}{z^2} + \frac{1}{z^2} G_{\mathbb{X}+\mathbb{Y}}(z) \right)$ for all $z \in \mathbb{C}^+$, where

$$A(z) = \frac{1}{\omega_2(z)} + \frac{\gamma}{\omega_2(z) G_{\mathbb{X}+\mathbb{Y}}(z)}, \quad B(z) = \frac{\varphi(\mathbb{Y}^{-2}) - \gamma A(z)}{\omega_2(z)}.$$

that allows us to determine distributions of $\mathbb{X}$ and $\mathbb{Y}$.

The distribution of $\mathbb{Y}$

- One can show that equations (1), (2), (3) imply

$$\omega_2(z) = \frac{d^2(\gamma + G_{\mathbb{X}+\mathbb{Y}}(z))}{G_{\mathbb{X}+\mathbb{Y}}(z)\left(d^2\gamma - (h-d^2)G_{\mathbb{X}+\mathbb{Y}}(z)\right)}.$$

- Since $G_{\mathbb{X}+\mathbb{Y}}(z) = G_{\mathbb{Y}}(\omega_2(z))$ this allows us to determine $G_{\mathbb{Y}}^{-1}(z)$ and thus

$$r_{\mathbb{Y}}(z) = G_{\mathbb{Y}}^{-1}(z) - \frac{1}{z} = \frac{h}{d^2\gamma - (h-d^2)z}.$$

- Consequently $\mathbb{Y}$ has the free Poisson distribution $\nu\left(\frac{h}{h-d^2}, \frac{h-d^2}{d^2\gamma}\right)$.

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The distribution of $\mathbb{X}$

- Equations (1), (2), (3) imply that $G = G_{\mathbb{X}+\mathbb{Y}}(z)$ satisfies

$$(h - d^2)z^2G^2 - d^2(\gamma z^2 - z - d)G + d^2(\gamma z + d\beta) = 0.$$

- The Cauchy-Stieltjes transform of the free-GIG distribution $\mu = \mu(\lambda, \alpha, \beta)$ is equal

$$G_\mu(z) = \frac{\alpha z^2 - (\lambda - 1)z - \beta - (\alpha z + \frac{\beta}{\sqrt{ab}})\sqrt{(z-a)(z-b)}}{2z^2},$$

and satisfies

$$z^2G_\mu^2 - (\alpha z^2 - (\lambda - 1)z - \beta)G_\mu + \alpha z + \delta = 0.$$

where δ depends on α, β, λ.

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The proof follows now from the following proposition

Szpojankowski [8]

- Suppose the function $G = G(z)$ satisfies the following equation

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  for some $\alpha, \beta, \delta > 0$ and $\lambda \in \mathbb{R}$. If G is the the Cauchy-Stieltjes transform of a positive random variable $\mathbb{X}$, then δ is uniquely determined by α, β, λ and $\mathbb{X}$ has the free-GIG distribution $\mu(\lambda, \alpha, \beta)$

- Let $\mathbb{X}$ and $\mathbb{Y}$ be self-adjoint random variables such that $\mathbb{X}$ has the free-GIG distribution $\mu(-\lambda, \alpha, \beta)$ and the distribution of $\mathbb{Y}$ is free-Poisson $\nu(\lambda, 1/\alpha)$. Then the distribution of $\mathbb{X} + \mathbb{Y}$ is free-GIG $\mu(\lambda, \alpha, \beta)$.

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Thank you for your attention.

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