Roots of Polynomials and Free Convolution

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UNIVERSITY of WASHINGTON

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The hope is that maybe these two different perspectives might perhaps be mutually beneficial.

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If you are outside the convex hull, the charges 'push you away'.

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Sketch. The roots interlace, each root moves roughly $\pm n^{-1}$ under one step of differentiation.

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What about the roots of

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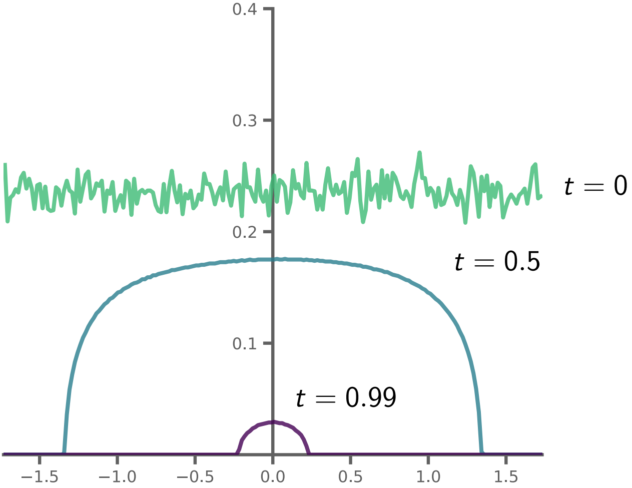

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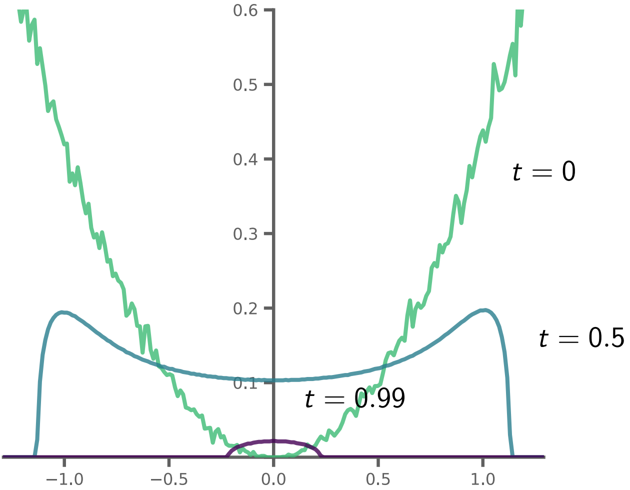

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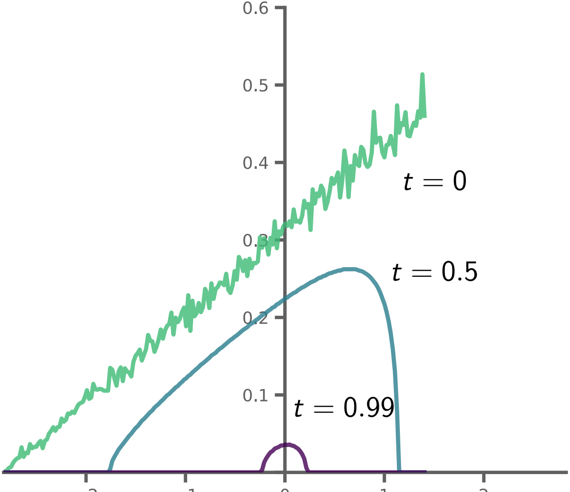

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3. The smallest gap grows under differentiation. Denoting the smallest gap of a polynomial p_n having n real roots $\{x_1, \dots, x_n\}$ by
$$G(p_n) = \min_{i \neq j} |x_i - x_j|,$$
we have (Riesz, Sz-Nagy, Walker, 1920s)
$$G(p_n') \geq G(p_n).$$

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What can one say about $u(t,x)$?

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Theorem (S. 2018)

If there exists a nice continuous evolution, then it is described by

$$\frac{\partial u}{\partial t} + \frac{1}{\pi}\frac{\partial}{\partial x}\arctan\left(\frac{Hu}{u}\right) = 0 \qquad \text{on } \operatorname{supp}(u)$$

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The argument is actually fun and I can give it in full. But before, let's explore this strange equation.

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Hermite Polynomials

Hermite polynomials $H_n : \mathbb{R} \to \mathbb{R}$ satisfy a nice recurrence relation

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This suggests that

$$u(t,x) = \frac{2}{\pi}\sqrt{1 - t - x^2} \cdot \chi_{|x| \le \sqrt{1-t}} \qquad \text{for } t \le 1$$

should be a solution of the PDE (and it is).

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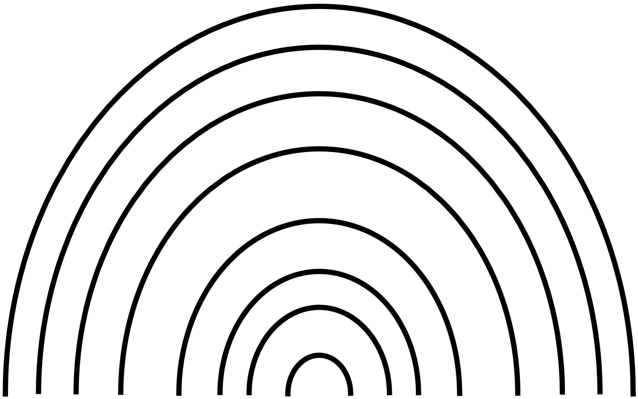

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The roots converge in distribution to the Marchenko-Pastur distribution

$$v(c,x) = \frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi x} \chi_{(x_-,x_+)} dx$$

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Indeed,

$$u_c(t,x) = v\left(\frac{c+t}{1-t}, \frac{x}{1-t}\right)$$

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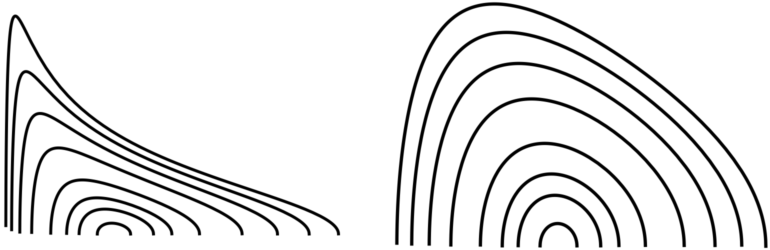

Figure: Marchenko-Pastur solutions $u_c(t,x)$: $c = 1$ (left) and $c = 15$ (right) shown for $t \in \{0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99\}$.

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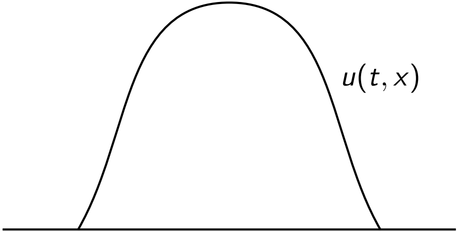

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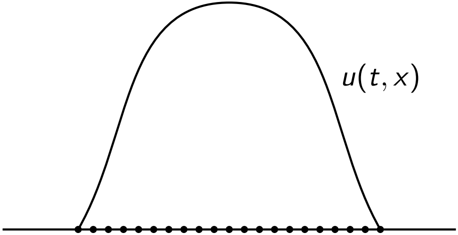

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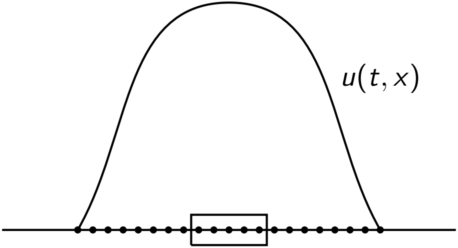

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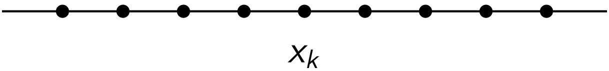

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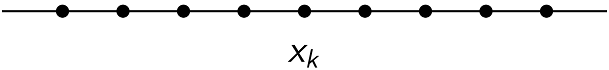

$$\sum_{k=1}^{n} \frac{1}{x - x_k} = \sum_{|x_k - x| \text{ large}} \frac{1}{x - x_k} + \sum_{|x_k - x| \text{ small}} \frac{1}{x - x_k}$$

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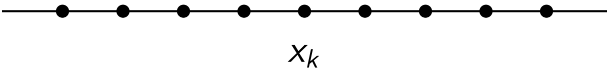

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$$\sum_{|x_k - x| \text{ large}} \frac{1}{x - x_k} \sim n \int_{\mathbb{R}} \frac{1}{x - y} \cdot u(t, y) dy = n \cdot [Hu](t, x).$$

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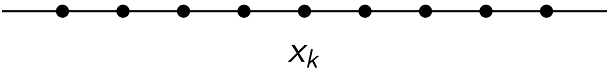

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It thus remains to understand the behavior of the local term.

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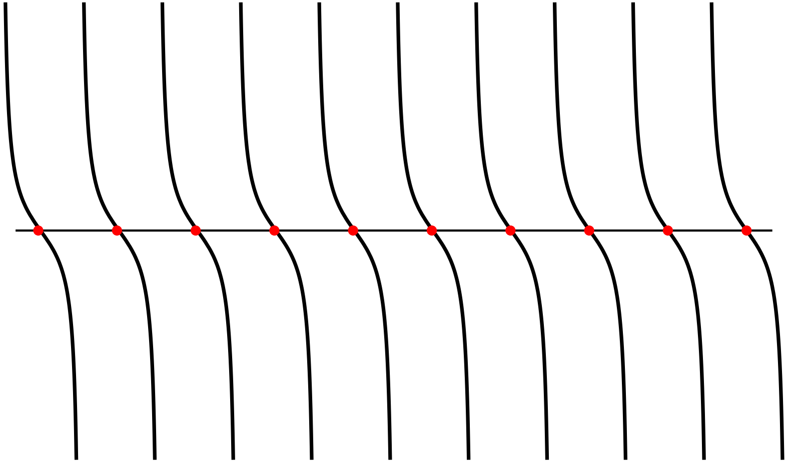
We are in luck: this sum has a closed-form expression due to Euler

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right) \qquad \text{for } x \in \mathbb{R} \setminus \mathbb{Z}.$$

The Local Field



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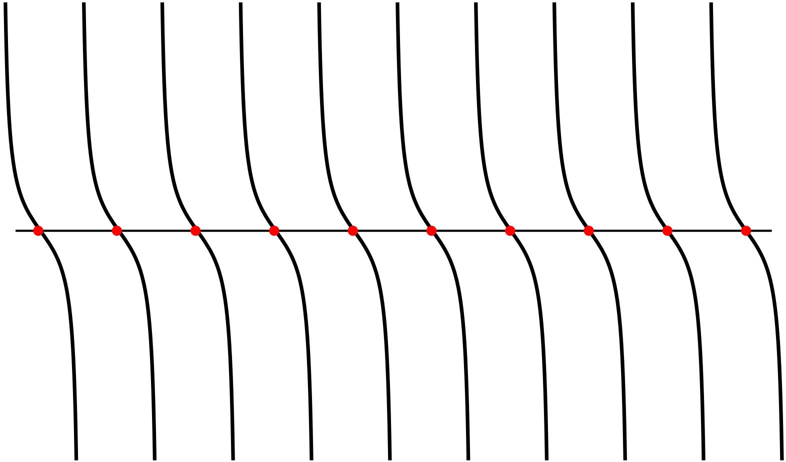

We can then predict the behavior of the roots of the derivative: they are in places where the local (near) field and the global (far) field cancel out. This leads to the desired equation.

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Rafael Granero-Belinchon then studied

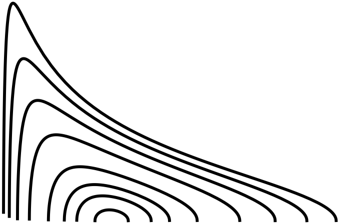
$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\arctan\left(\frac{Hu}{u}\right) = 0 \qquad \text{on } \mathbb{S}^1$$

where

$$Hu(x) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{u(y)}{\tan\left(\frac{x-y}{2}\right)}dy.$$

and proved global solutions for $u(0,x) \in H^2(\mathbb{S}^1)$.

A Fast Numerical Algorithm



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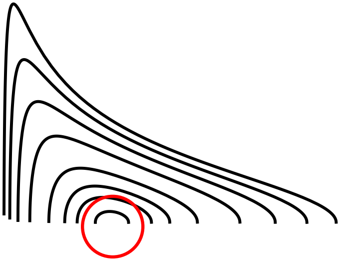

Jeremy Hoskins (U Chicago) used the electrostatic interpretation to produce an algorithm that can compute all derivatives of polynomials up to degree ~ 100.000. **Semicircles at the end**.

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Let X be a random variable on $\mathbb{R}$ such that all moments are finite and $\mathbb{E}X = 0$ as well as $\mathbb{V}X = 1$.

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Remarks.

1. The roots of the Hermite polynomial have a semicircle density.
2. If $x_1, x_2, \dots, x_n \sim X$, then

$$\frac{x_1 + \cdots + x_n}{\sqrt{n}} \sim \mathcal{N}(0,1)$$

   and the mean of the roots is preserved under differentiation (that's why there is a random shift).

Sep 3, 2020

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Fractional free convolution powers

Dimitri Shlyakhtenko, Terence Tao

The extension $k \mapsto \mu^{\boxplus k}$ of the concept of a free convolution power to the case of non-integer

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(where we use the branch of arctan taking values in $[0, \pi]$) and thus by the change of variables $k = 1/s$ and abbreviating $f := f_{1/s}$,

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The same PDE in a supposedly different context is presumably not a coincidence.

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Under some reasonable assumptions

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$u(t, x)$ should be close to a semicircle for t close to 1. That's exactly the Theorem that Jeremy and I proved for polynomials.

Sep 3, 2020

Sep 4, 2020

Sep 4, 2020

[Submitted on 4 Sep 2020]

Universal objects of the infinite beta random matrix theory

Vadim Gorin, Victor Kleptsyn

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which proves that, in a certain setting, the crystallization assumption for roots is justified in the bulk

Oct 27, 2020

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JEREMY HOSKINS AND ZAKHAR KABLUCHKO

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Dec 16, 2020

THE FLOW OF POLYNOMIAL ROOTS UNDER DIFFERENTIATION

ALEXANDER KISELEV AND CHANGHUI TAN

ABSTRACT. The question about the behavior of gaps between zeros of polynomials under differentiation is classical and goes back to Marcel Riesz. In this paper, we analyze a nonlocal nonlinear partial differential equation formally derived by Stefan Steinerberger [55] to model dynamics of roots of polynomials under differentiation. Interestingly, the same equation has also been recently obtained formally by Dimitri Shlyakhtenko and Terence Tao as the evolution equation for free fractional convolution of a measure [51] - an object in free probability that is also related to minor processes for random matrices. The partial differential equation bears striking resemblance to hydrodynamic models used to describe the collective behavior of agents (such as birds, fish or robots) in mathematical biology. We consider periodic setting and show global regularity and exponential in time convergence to uniform density for solutions corresponding to strictly positive smooth initial data. In the second part of the paper we connect

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- $(-\Delta)^{1/2}$ appears quite naturally!

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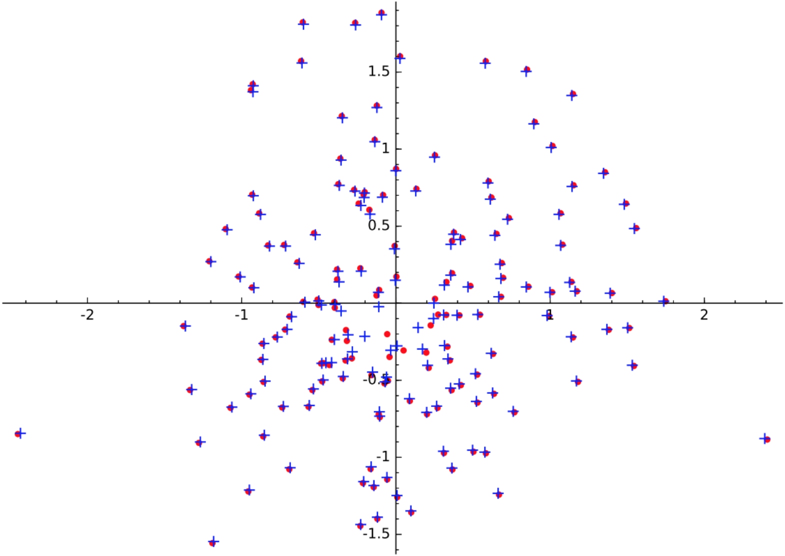
- What about the complex case?
- There is a PDE but it's more complicated (essentially a complex Burger's equation), the radial case becomes a really nice one-dimensional transport equation (joint with Sean O'Rourke)



picture from O'Rourke and Williams (2018)

A Nonlocal Transport Equation

Sean O'Rourke and I tried to see whether the equation simplifies if we assume that the initial distribution is radial around the origin.

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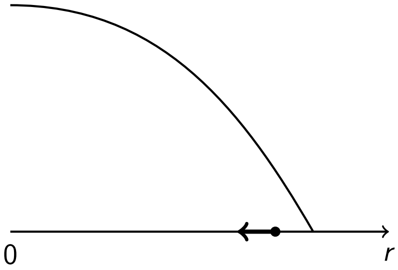

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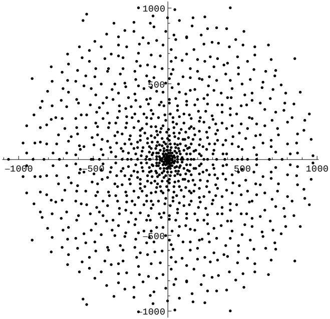
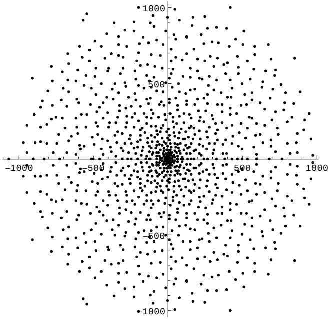
This corresponds to **Random Taylor Polynomials**.





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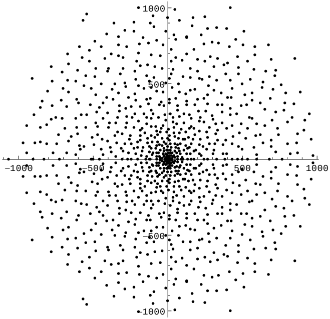
$$p_n = \sum_{k=0}^{n} \gamma_k \frac{z^k}{k!},$$

where $\gamma_k \sim \mathcal{N}(0,1)$.



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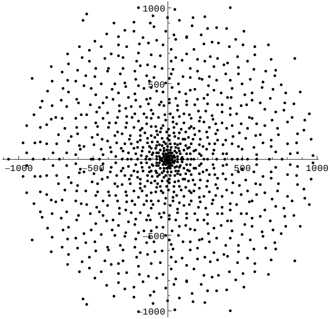

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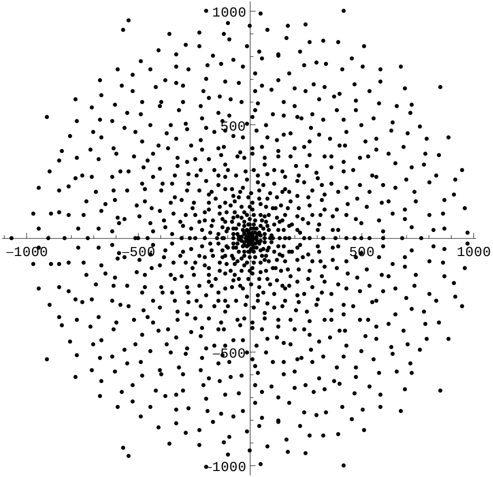
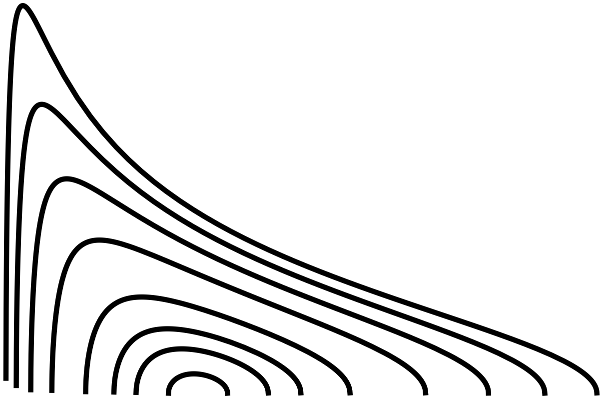
where $\gamma_k \sim \mathcal{N}(0,1)$. They are preserved under differentiation.

Theorem (Kabluchko & Zaporozhets)

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{z_k n^{-1}} \to \frac{\chi_{|z|\le 1}}{2\pi|z|} \qquad \text{as } n \to \infty.$$



Kabluchko & Hoskins give many more examples of solutions.



Thank you!