

# Non-Commutative Stochastic Processes and Bi-Free Probability

$$\text{NCSP} \quad t \mapsto X_t \\ \in T \quad \in (M, \tau)$$

## Free (Centred) Gaussian Processes

- $\mathcal{H}$  a real Hilbert space  $\leadsto \mathcal{H} \subset \mathbb{C}$
- $T, (f_t)_{t \in T}$  where  $f_t \in \mathcal{H}$ .
- $X_t = \ell(f_t) + \ell^*(f_t)$
- Bożejko, Kümmerer, Speicher, only depend on  $c(t_1, t_2) = \langle f_{t_1}, f_{t_2} \rangle \leadsto$  Markov Process
- Free Brownian Motion  
 $T = [0, \infty), \quad c(t_1, t_2) = \min(t_1, t_2)$
- Free Brownian Bridge  
 $T = [0, 1], \quad c(t_1, t_2) = t_1(1-t_2)$

For  $l < r$ , the expectation of  $W^*(X_r)$  onto  $W^*(\{X_t, t \leq l\})$  lies in  $W^*(X_l)$ .

$$W^*(X_r) \cong L_\infty(\mu_r) \quad W^*(X_e) \cong L_\infty(\mu_e)$$

$$E: W^*(X_r) \rightarrow W^*(X_e)$$

Transition Operator  $K_{e,r}: L_\infty(\mu_r) \rightarrow L_\infty(\mu_e)$  s.t.

$$E(h(X_r)) = K_{e,r}(h)(X_e) \quad \forall h \text{ Borel.}$$

Free Gaussian Process Transition Ops

$$K_{e,r}(X, dy) = \frac{1}{2\pi\lambda_r^2} \frac{(1-\lambda_{e,r}^2) \sqrt{4\lambda_r^2 - y^2}}{(1-\lambda_{e,r}^2)^2 - \lambda_{e,r}(1+\lambda_{e,r}^2)\left(\frac{x}{\lambda_e}\right)\left(\frac{y}{\lambda_r}\right)} dy$$

$$\lambda_r = \sqrt{C(r,r)} \quad + \lambda_{e,r}^2 \left( \left(\frac{x}{\lambda_e}\right)^2 + \left(\frac{y}{\lambda_r}\right)^2 \right)$$

$$\lambda_{e,r} = \frac{C(e,r)}{\lambda_e \lambda_r}$$

$$K_{e,r}(h)(x) = \int h(y) K_{e,r}(x, dy)$$

Bifree Central Limit distribution (Huang, Wang)

$$f(x, y) dx dy = \frac{1}{2\pi\lambda_e^2} \sqrt{4\lambda_e^2 - x^2} K_{e,r}(x, dy) dx$$

$$X_e \sim C(e, e)$$

$$X_r \sim C(r, r) \quad \langle X_e, X_r \rangle = C(e, r)$$

$X_e, X_r$  self-adjoint in  $(m, \tau)$

We want to understand  $E: W^*(X_r) \rightarrow W^*(X_e)$

$$\tau(E(X_r^m) X_e^n) = \tau(X_e^n X_r^m) = \langle X_e^n X_r^m, \xi \rangle$$

$$\text{GNS}_{(M, \tau)} \rightsquigarrow L_2(M, \tau)$$

$\xi$  unit vector  $\xi = 1_M$

For  $S \in M$ ,  $L(S)$  left action of  $S$  on  $L_2(M, \tau)$   
 $R(S)$  right action of  $S$  on  $L_2(M, \tau)$

$$\tau(X_e^n X_r^m) = \langle L(X_e^n) R(X_r^m) \xi, \xi \rangle$$

$\rightsquigarrow$  understand  $(L(X_e), R(X_r))$   $\rightsquigarrow$   $\mu_{e,r}$  on  $\mathbb{R}^2$   
commute

$$\xrightarrow{\text{Luck!}} \mu_{e,r} = f_{e,r}(x,y) dx dy$$

$$(\text{Ker}_r(h))(x) = \int h(y) \text{Ker}_r(x, dy) \text{ where}$$

$$\text{Ker}_r(x, dy) = \frac{f_{e,r}(x,y)}{f_e(x)} dy$$

$$\uparrow f_e(x) = \int f_{e,r}(x,y) dy$$

## Example Free Poisson Process

- projection valued process:  $I \mapsto P_I$  s.t.  $P_I \llcorner P_J$
  - if  $I, J \subseteq [0,1]$  disjoint then  $P_I, P_J$  are orthogonal, and  $P_I + P_J = P_{I \cup J}$
  - $\tau(P_I) = |I|$ ,
  - Take  $S$  semi circle free from  $\{P_I \mid I \subseteq [0,1]\}$ .
  - $I \mapsto S P_I S$ .
  - $X_t = S P_{[0,t]} S$
  - $(L(X_e), R(X_r)) \rightsquigarrow \mu_{e,r}$  lcr.  
is the bi-free Poisson distribution with rate  $\lambda=r$  and jump size  $\nu = \frac{r}{r} \delta_{(1,0)} + \frac{(r-2)}{r} \delta_{(1,1)}$
- 

## Bi-Free Transform

### Two-variable Green's function

$$G_{x,y}(z,w) = \tau((z-x)^{-1} (w-y)^{-1})$$

$$= \int_{\mathbb{R}^2} \frac{1}{z-x} \frac{1}{w-y} d\mu_{x,y}(x,y)$$

- If  $d\mu_{x,y}(x,y) = f(x,y) dx dy$  then

$$f(x,y) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi^2} \operatorname{Im} \left( \frac{G_{x,y}(x+i\varepsilon, y+i\varepsilon) - G_{x,y}(x+i\varepsilon, y-i\varepsilon)}{2i} \right)$$

$$\bullet R_{X,Y}(z,w) = \sum_{\substack{n,m \geq 0 \\ m+n \geq 1}} k_{n,m}(X,Y) z^n w^m$$

$$= 1 + z R_X(z) + w R_Y(w) - \frac{zw}{G_{X,Y}(k_X(z), k_Y(w))}$$

$$\tilde{R}_{X,Y}(z,w) = 1 - \frac{zw}{G_{X,Y}(k_X(z), k_Y(w))}$$

$$= \sum_{n,m \geq 1} k_{n,m}(X,Y) z^n w^m$$

$(X,Y)$  bi-free from  $(X',Y')$ , then

$$\tilde{R}_{X+X', Y+Y'}(z,w) = \tilde{R}_{X,Y}(z,w) + \tilde{R}_{X',Y'}(z,w).$$

$$\tilde{R}_{X,Y}(z,w) = Czw$$

Bi-free  
central limit

Biane 1998

Defn free additive increments,  $\forall t_1 < t_2 < \dots < t_n$ ,  
 $X_{t_1}, X_{t_2-t_1}, X_{t_3-t_2}, \dots, X_{t_n-t_{n-1}}$  are free

$r \geq 2$   
 $X_e, X_r$  with freely additive increments  
 $X = X_e, Y = X_r - X_e$

Thm (S, 2022) If  $X, Y$  free independent, then

$$G_{L(X), R(X+Y)}(z, w) = - \frac{G_X(z) - G_{X+Y}(w)}{z - K_X(G_{X+Y}(w))}$$

Proof

$X, Y$  free  $\rightsquigarrow$   $(L(X), R(X))$  and  $(L(0), R(Y))$   
 bi-free

$$\tilde{R}_{L(X), R(X+Y)}(z, w) = \tilde{R}_{L(X), R(X)}(z, w) + \tilde{R}_{L(0), R(Y)}(z, w)$$

$$= \sum_{n, m \geq 1} K_{n, m}(L(X), R(X)) z^n w^m$$

$$= \sum_{n, m \geq 1} K_{n+m}(X) z^n w^m$$

$$= zw \frac{R_X(z) - R_X(w)}{z - w} \quad \begin{matrix} n+m=k \\ \frac{z^{k+1} - w^{k+1}}{z - w} \end{matrix}$$

Example free Cauchy Processes

$$\mu_+(dx) = \frac{1}{\pi} \frac{t}{x^2 + t^2} dx$$

$$\rightsquigarrow G_{X_t}(z) = \begin{cases} \frac{1}{z+it} & \text{Im}(z) > 0 \\ \frac{1}{z-it} & \text{Im}(z) < 0 \end{cases}$$

$$\operatorname{Im}(z) > 0, \operatorname{Im}(w) > 0 \rightarrow G_{L(x_e), R(x_r)}(z, w) = \frac{1}{z + i\ell} \frac{1}{w + ir}$$

$$\operatorname{Im}(z) > 0, \operatorname{Im}(w) < 0 \rightarrow = \frac{(z-w) + i(\ell+r)}{(z+i\ell)(w-ir)(z-w+i(r-\ell))}$$

$$\dots$$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi^2} \operatorname{Im} \left( \frac{G(x+i\varepsilon, y+i\varepsilon) - G(x+i\varepsilon, y-i\varepsilon)}{2i} \right)$$

$$= \frac{1}{\pi^2} \frac{\ell}{x^2 + \ell^2} \frac{r-\ell}{(x-y)^2 + (r-\ell)^2}$$

$$K_{\ell, r}(x, dy) = \frac{1}{\pi} \frac{r-\ell}{(x-y)^2 + (r-\ell)^2} dy$$


---

- $(x_t)_{t \in T}$  in  $(M, \tau_M)$  and  $(y_t)_{t \in T}$  in  $(N, \tau_N)$

- $(M * N, \tau_M * \tau_N) \rightarrow (x_t), (y_t)$  in sncps  
s.t.  $\{x_t, t \in T\}$  free from  $\{y_t, t \in T\}$

- $(x_t + y_t)_{t \in T}$

$(x_e, x_r)$

Transition ops

$(y_e, y_r)$

$\Downarrow$

$(L(x_e), R(x_r))$

Distribution

$(L(y_e), R(y_r))$

Bi-free

Distribution of  $(L(x_t + y_t), R(x_r + y_r))$

↓

Transition ops of  $(x_t + y_t)_{t \in T}$

Unitaries

$\tau(u) \neq 0$

$(u_t)_{t \in T}, (v_t)_{t \in T}$  free

$\leadsto (u_t v_t)_{t \in T}$

$\leadsto (L(u_t v_t), R(u_r v_r))$

$(L(u_t) L(v_t), R(v_r) R(u_r))$

of multiplication.

Transformations

•  $\Psi_u(z) = \int_{\pi} \frac{zs}{1-zs} da(s) \quad z \in \mathbb{C} \setminus \mathbb{T}$

• If  $da(s) = f(s) ds$ ,  
 $\operatorname{Re}(2\Psi_u(z) + 1) = \int_{\pi} \operatorname{Re}\left(\frac{1+zs}{1-zs}\right) da(s)$

→ poisson integral of  $da(\frac{1}{s})$ .

$$\bullet m_u(z) = \frac{\psi_u(z)}{1 + \psi_u(z)}$$

$$\bullet S\text{-transform } S_u\left(\frac{z}{1+z}\right) = \frac{1}{z} m_u^{-1}(z).$$

$$\bullet \psi_{u,v}(z,w) = \int_{\pi^2} \frac{zs}{1-zs} \frac{wt}{1-wt} da_{u,v}(s,t)$$

$$\bullet g_{u,v}(z,w) = 4\psi_{u,v}(z,w) + 2\psi_u(z) + 2\psi_v(w) + 1$$
$$= \int_{\pi^2} \frac{1+2s}{1-2s} \frac{1+wt}{1-wt} da_{u,v}(s,t)$$

$$\bullet \operatorname{Re}\left(\frac{g(z,w) - g\left(\frac{1}{z}, w\right)}{2}\right) = \int_{\pi^2} \operatorname{Re}\left(\frac{1+2s}{1-2s}\right) \operatorname{Re}\left(\frac{1+wt}{1-wt}\right) da(s,t)$$

Poisson integral of  $da\left(\frac{1}{s}, \frac{1}{t}\right)$ .

$$\bullet H_{u,v}(z,w) = 1 + \psi_u(z) + \psi_v(w) + \psi_{u,v}(z,w)$$
$$= \int_{\pi^2} \frac{1}{(1-zs)(1-wt)} da(s,t).$$

• OP bi-free partial S-Transform

Huang, Wang :

$$S_{u,v}^{\text{op}}(z,w) = \frac{w(z+1)}{z(w+1)} \frac{H_{u,v}(\Psi_u^{-1}(z), \Psi_v^{-1}(w)) - (w+1)}{H_{u,v}(\Psi_u^{-1}(z), \Psi_v^{-1}(w)) - (z+1)}$$

- $(y,v)$  bi-free from  $(u',v')$ , then

$$S_{uu',v'v}^{\text{op}}(z,w) = S_{y,v}^{\text{op}}(z,w) S_{u',v'}^{\text{op}}(z,w)$$

- If  $K_{u,v}(z,w) = \sum_{n,m \geq 1} \kappa_{n,m}(u,v) z^n w^m$

$$\text{then } S_{u,v}^{\text{op}}(z,w) = \frac{1 + \frac{1}{z} K_{u,v}(z S_u(z), w S_v(w))}{1 + \frac{1}{w} K_{u,v}(z S_u(z), w S_v(w))}$$

Defn (Biane) (left) multiplicatively free processes  
 $\forall t_1 < t_2 < \dots < t_k, \quad U_{t_1}, \underline{U_{t_2} U_{t_1}^{-1}}, \dots, U_{t_k} U_{t_{k-1}}^{-1}$   
 are free.

$$U_e, U_r \rightsquigarrow (L(U_e), R(U_r))$$

$U_e, U_r U_e^{-1}$  are free

$$(L(u), R(vu)) \quad \text{if } u = U_e, v = U_r U_e^{-1}$$

Thm. If  $U$  and  $V$  are free, then

$$H(L(U), R(V)) (z, w) = 1 + \frac{z \Psi_U(z) - \Psi_U^{-1}(\Psi_{VU}(w)) \Psi_{VU}(w)}{z - \Psi_U^{-1}(\Psi_{VU}(w))}$$

Proof

$U, V$  free  $\rightarrow (L(U), R(U)), (L(V), R(V))$  bi-free.

$$S_{L(U), R(V)}^{\text{op}} (z, w) = S_{L(U), R(V)}^{\text{op}} (z, w) S_{L(U), R(U)}^{\text{op}} (z, w)$$

$L(U) L(U) R(U) R(V)$

Example If  $\mu_t(ds) = \frac{1 - e^{-2t}}{|s - e^{-t}|^2} ds$   $\leftarrow$  unitary free Levy process

$$\dots \kappa_{e,r}(s, dt) = \frac{1 - e^{2t(r-r')}}{|s^{-1}t - e^{e^{-r}}|^2} dt.$$