MEASURABILITY, SPECTRAL DENSITIES AND HYPERTRACES IN NONCOMMUTATIVE GEOMETRY

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Spectral density for a compact Riemannian manifold

This is the founding example of Non Commutative Geometry :

H. Weyl asymptotic formula for the spectrum of the Laplacian

$$\sharp\{eigenvalues \leq x\} \sim C_d \, x^{-d/2} \, vol(M)$$

The same for the spectrum of the module of the Dirac operator

$$\sharp\{eigenvalues \leq x\} \sim C_d \, x^{-d} \, vol(M)$$

from which Alain Connes derives two singular trace formulas

A.C. volume formulas for the module of the Dirac operator

$$1/ \; Tr_\omega(|D|^{-d}) = C_d . vol(M)$$

$$2/ \; Tr_\omega(f \, |D|^{-d}) = C_d \int_M f(x) dvol(x) \, , f \in C(M)$$

In other words, the asymptotic behavior of the spectrum of the Dirac operator determines both the volume of the manifold and the volume form.

D.V.V.'s Dirac operator associated with a filtration

D. Voiculescu, On the existence of quasicentral approximate units relative to normed ideals. I., *J. Funct. Anal.* 91 (1990), no. 1, 1-36. Cf. Proposition 5.1.

The spectral triple $(\mathcal{A}, h, D)$ is constructed as follows :
Increasing sequence $h_0 \subset h_1 \subset \cdots \subset h_n \subset \cdots$ of finite dim. subspaces (with $\overline{\bigcup h_n} = h$)
$\rightarrow$ increasing sequence $P_0 \leq P_1 \leq \cdots \leq P_n \leq \cdots$ of finite rank orth. projections (with $\lim_n P_n = I_H$)

The algebra $\mathcal{A}$ is $\mathcal{A} = \bigcup_k \{T \in \mathcal{B}(H) \ : T(h_n) \cup T^*(h_n) \subset h_{n+k} \, , \, k \in \mathbb{N}\}$

The Dirac operator is $D = \sum_n (I_H - P_n)$.
It has spectrum $\mathbb{N}$, and the eigenvalue k has multiplicity $rk(P_k) - rk(P_{k-1})$.

The questions could be :

Which density for this spectral triple ?
Which function of D is a candidate for being a density ?
Which volume form does it provide ?

The more general question is :

> Is there a nice notion of density for general spectral triples ?
> Is there a nice volume form associated with it ?

The answer will be : often.

The workframe will be a slight generalization of the notion of spectral triple, where we allow D to have infinite dimensional kernel.
(Cf. [CS].Fredholm Modules on P.C.F. Self-Similar Fractals and their Conformal Geometry, Comm. Math. Phys. 286 (2009), no. 2
or [CGIS]. Spectral triples for the Sierpinski Gasket, J. Funct. Anal., 266 (2014) 4809-4869)

From now on, we consider a general operator L nonnegative selfadjoint on a Hilbert space H with discrete spectrum away from its kernel.

Later, we apply the results to $L = |D|$, with D the Dirac operator of a general spectral triple.

Operators with discrete spectrum away from their kernels

From now on, L will be an operator on H nonnegative selfadjoint,.

P_0 is the orthogonal projection on $Ker(L)$.

Assumption : L has discrete spectrum away from its kernel

which means : P_0 can have finite or infinite range
while $(I - P_0)L$ has discrete spectrum on $(I - P_0)(H)$.

By convention, for L positive self adjoint and $f : \mathbb{R} \to \mathbb{R}$, we set

$$f(L) = 0.P_0 + f\big((I - P_0)L\big)$$

With this convention, L has discrete spectrum away from its kernel iff L^{-1} is a compact operator.

A digression on Dirac's having infinite dimensional kernels

D densely defined self adjoint on H

P_0 orth. proj. on $Ker(D)$

$D(I - P_0)$ has discrete spectrum in $Ker(D)^\perp$

By convention D^{-1} is 0 on $Ker(D)$, so that $DD^{-1} = D^{-1}D = I - P_0$

For $a \in \mathcal{A}$ (i.e. $[a, D]$ is bounded), one checks easily

$$\begin{aligned}[P_0, a] &= P_0[a, D]D^{-1} + D^{-1}[a, D]P_0 \\ &= bounded \times D^{-1} + D^{-1} \times bounded\end{aligned}$$

So that $[P_0, a]$ is a compact operator and we have an estimate on the decay for its characteristic values :

$$\mu_n([P_0, a]) = O\big(\mu_n(|D|^{-1})\big)$$

Enumeration of the eigenvalues

There are two ways for describing the spectrum of L :

first way

$$sp(L)\backslash\{0\} = \{0 < \lambda_1 \leq \lambda_1 \leq \ldots \leq \lambda_n \leq \ldots\} \quad \lim_n \lambda_n = +\infty$$

with repetition according to the multiplicity.

second way

$$sp(L)\backslash\{0\} = \{0 < \widetilde{\lambda}_1 < \widetilde{\lambda}_2 < \ldots < \widetilde{\lambda}_k < \ldots\} \quad \lim_k \widetilde{\lambda}_k = +\infty$$

with *multiplicity of* $\widetilde{\lambda}_k = m_k$.

They correspond through the formula

$$\lambda_n = \widetilde{\lambda}_k \text{ for } \sum_{j=1}^{k} m_j \leq n < \sum_{j=1}^{k+1} m_j$$

or

$$\lambda_n = \widetilde{\lambda}_k \text{ for } M_k \leq n < M_{k+1}$$

cumulated multiplicities :

$$M_k = \sum_{j=1}^{k} m_j = Tr(E_L(0, \widetilde{\lambda}_k])$$

D.V.V's example

If $L = D = \sum_k (I - P_k)$ is the Dirac operator associated with a filtration of a Hilbert space

$$\widetilde{\lambda}_k = k, \quad k \in \mathbb{N}^*$$

$$m_k = rk(P_k - P_{k-1})$$

$$M_k = rk(P_k - P_0)$$

$$\lambda_n = \widetilde{\lambda}_k \text{ for } M_k \leq n < M_{k+1}$$

First part

Density associated with L

A probabilistic observation

Let μ be a Radon measure on $\mathbb{R}_+$ with repartition function $\mathcal{F}_\mu$

$$\mathcal{F}_\mu(t) = \mu([0,t])$$

First case : μ is a diffuse probability.

Then $\mathcal{F}_\mu(\mu)$ is the uniform probability measure on $[0,1]$.

Second case : μ is a discrete sum of Dirac measures : $\mu = \sum_k \delta_{u_k}$ with $u_k \uparrow \infty$.

Then $\mathcal{F}_\mu(\mu) = \sum_{k\in\mathbb{N}} \delta_k$ the counting function on the integers .

In both cases $\mathcal{F}_\mu(\mu)$ does not depend on μ.

Third case : $\mu = \sum_k m_k \delta_{u_k}$ discrete with weights.

Then what can we say about $\mathcal{F}_\mu(\mu)$?

We shall try to answer in the more general workframe of spectral measures.

The counting function of L

The counting function

$$F_L(x) = \sharp\{n \in \mathbb{N}^* \mid \lambda_n \leq x\} = Tr(E_L(0,x])$$

First observation

$$F_L(\widetilde{\lambda}_k) = M_k \quad F_L(x) = M_k \,,\ \widetilde{\lambda}_k \leq x < \widetilde{\lambda}_{k+1}$$

Second observation

1/ F_L is right continuous, with left limits $F_L^-(x) = \lim_{t \uparrow x} F_L(t)$.

2/ $\limsup_{x \to +\infty} \dfrac{F_L(x)}{F_L^-(x)} = \limsup_{k \to \infty} \dfrac{M_{k+1}}{M_k}$.

The associated density

The density associated with L will be

$$\rho(L) = F_L(L)^{-1}$$

Third observation (on the characteristic values)

$$\mu_n(\rho(L)) = \frac{1}{M_k} \text{ when } M_k \leq n < M_{k+1}$$
$$\text{and} \quad \frac{M_{k-1}}{M_k}\frac{1}{n} \leq \mu_n(\rho(L)) \leq \frac{1}{n}$$

Remark : if all the multiplicities were equal to 1, one would have

$$\mu_n(\rho(L)) = 1/n, n \in \mathbb{N}^* ,$$

which in turn would imply

$$Tr_\omega(\rho(L)) = 1 \text{ for any singular trace } Tr_\omega$$

$$Tr(\rho(L)^s) = \zeta(s), s \in \mathbb{C}, \mathcal{R}e(s) > 1 \text{ with } \zeta \text{ the Riemann zeta-function.}$$

Some remarks

1/ The spectral distribution of $\rho(L)$
– does not depend on the precise location of the eigenvalues of L
– depends only on the sequence (m_k) of multiplicities.

2/ Similarly, $\rho(L)$
– does not depend on the precise location of the eigenvalues of L
– depends only on the sequence of eigenspaces.

3/ If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, $\lim_{x\to+\infty} f(x) = +\infty$, then

$$\rho\big(f(L)\big) = \rho\big(L\big)\,.$$

For instance, for a Riemannian manifold

$$\rho(|D|) = \rho(-\Delta)\,.$$

4/ Dan's example :

$$L = \textstyle\sum_{k\geq 1}(I - P_k) \qquad \rho(L) = \displaystyle\sum_{k\geq 1} \frac{1}{rk(P_k) - rk(P_0)}(P_k - P_{k-1})$$

5/ Example : G is a finitely generated group, B_k is the ball of radius k with cardinality $|B_k|$, L is the operator of multiplication by ℓ in $\ell^2(G)$

$$\rho(L) = \text{multiplication by } \sum_{k\geq 1} \frac{1}{|B_k| - 1} 1_{B_k \setminus B_{k-1}}$$

Computation of $Tr_\omega(\rho(L))$

Remind: $$\frac{M_{k-1}}{M_k}\frac{1}{n} \leq \mu_n(\rho(L)) \leq \frac{1}{n}$$

Consequence 1. Per ogni Dixmier ultrafilter ω

1/ $\rho(L) \in \mathcal{L}^{1,\infty}(H)$ and $Tr_\omega(\rho(L) \leq 1$

2/ If $\lim_k \frac{M_{k+1}}{M_k} = 1$ $\left(\text{or equivalenly } \lim_k \frac{m_k}{M_k} = 0\right)$, then $Tr_\omega(\rho(L)) = 1$

Consequence 2. Per ogni Dixmier ultrafilter ω

3/ If $\limsup_k \frac{M_{k+1}}{M_k} = c > 1$, then $Tr_\omega(\rho(L)) > 1/c$ (hence $\neq 0$)

4/ If $\lim_k \frac{M_{k+1}}{M_k} = c > 1$, then $Tr_\omega(\rho(L)) = \frac{c-1}{c\,\mathrm{Log}\,c}$

In cases 2 and 4 (i.e. $\lim_k M_{k+1}/M_k$ exists), the value of $Tr_\omega(\rho(L))$ does not depend on ω, i.e. $\rho(L)$ is measurable in the sense of A. Connes.

The Dixmier trace

ω is an ultrafilter on $\mathbb{N}$ with suitable properties.

T compact positive has eigenvalues $\mu_1(T) \geq \mu_2(T) \geq \cdots$ with $\mu_n(T) \to 0$ as $n \to \infty$
(decreasing order with repetition according to multiplicity).

$$Tr_\omega = \omega - lim_{n\to\infty} \frac{1}{Log(n)} \sum_{j=1}^{n} \mu_j(T)$$

extends as a trace on $\mathcal{K}(H)$.

Asymptotic continuity of the counting function

Asymptotic continuity

The following assertions are equivalent

1/ $\lim_k \frac{M_{k+1}}{M_k} = 1$

1bis/ $\lim_k \frac{m_k}{M_k} = 0$

2/ There exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous such that $\varphi(x) \sim F_L(x), x \to +\infty$.

In this case, we say that F_L is asymptotically continuous.

Remark 1

$$\text{In this case, we have } Tr_\omega(\rho(L) = Tr_\omega(\varphi(L)^{-1}) = 1.$$

Remark 2

$$\lim_k \frac{M_{k+1}}{M_k} = 1 \Rightarrow \lim_k \sqrt[k]{M_k} = 1 \text{ and } \lim_{k\to\infty} e^{-\beta k} M_k = 0 \ \forall \beta > 0$$

Further examples

Further examples include
– Some Euclidean domains having infinite volume, but however the Laplace-Beltrami operator has discrete spectrum.
For instance $\Gamma = \{(x,y) \in \mathbb{R}^2 \, , \, |xy| \leq 1\}$ for which

$$F_\Delta(x) \sim \frac{1}{\pi} x \, Log(x) \qquad \text{[B.Simon]}$$

$$\rho(\Delta) \sim \frac{1}{\pi} \frac{I}{\Delta \, Log(\Delta)} \text{ with } Tr_\omega\left(\frac{I}{\Delta \, Log(\Delta)}\right) = \pi$$

– Kigami harmonic structures on p.c.f. fractal sets :

$$F_L(x) \sim c \cdot x^{d_s/2} \qquad \text{[Kigami-Lapidus]}$$

$$Tr_\omega(L^{-d_s/2}) = c.$$

etc.

Second part

The zeta function

The ζ function of L

$$\zeta_L(s) = Tr(\rho(L)^s) = Tr(F_L(L)^{-s}) \,, \quad s \in \mathbb{C} \quad Re(s) > 1$$

When F_L is asymptotically continuous, then

$$\lim_{s\in\mathbb{R}\,,\, s\downarrow 1} (s-1)\zeta_L(s) = 1 \,.$$

Observation :
Let us suppose that every eigenvalue $\widetilde{\lambda}_k$ is simple ($m_k = 1 \ \forall k$). Then ζ_L coincides with the Riemann ζ function and hence extends meromorphically to the whole complex plane. It has a unique pole pole at $s = 1$ with residue 1.

First criterion for a meromorphic extension

if there exists $\alpha \in (0,1)$ s.t. $\sum_k \frac{m_k^2}{M_k^{1+\alpha}} < +\infty$, then

ζ_L has a meromorphic extension to the upper half plane $Re(s) > \alpha$ with a unique pole at $s = 1$ and residue 1.

More criteria for a meromorphic extension

Second criterion

If there exists $\alpha \in (0,1)$ s.t. $m_k = O(M_k^\alpha)$, $k \to \infty$, then

ζ_L has a meromorphic extension to the upper half plane $Re(s) > \alpha$ with a unique pole at $s = 1$ and residue 1.

An equivalence

For $\alpha \in (0,1)$ the two conditions are equivalent :

(i) $m_k = O(M_k^\alpha)$, $k \to \infty$

(ii) $\exists\, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous s.t. $F_L(x) = \varphi(x) + O(\varphi(x)^\alpha)\ \ x \to +\infty$

Finally

Third criterion

If there exist $\alpha \in (0,1)$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous s.t.

$$F_L(x) = \varphi(x) + O(\varphi(x)^\alpha)\ \ x \to +\infty,$$

then

ζ_L has a meromorphic extension to the upper half plane $Re(s) > \alpha$ with a unique pole at $s = 1$ and residue 1.

First examples

L is the Friedrich extension of a pseudodiferential operator of order k, positive and symmetric with respect to the symplectic measure.
Weyl-Hormander estimate :

$$F_L(x) = C \cdot x^{n/k} + O(x^{(n-1)/k})$$

$$\to \zeta_L \text{ extends meromorphically to } \mathcal{R}e(s) > 1 - \frac{1}{n}.$$

Can be improved in some examples. For instance : for Δ the Laplace-Beltrami operator on the product of two spheres $S^2 \times S^2$

$$F_\Delta(x) = c \cdot x^2 + O(x^{4/3}) \text{ (M.Taylor)}$$

$$\to \zeta_\Delta \text{ extends meromorphically to } \mathcal{R}e(s) > 2/3.$$

Same result for $\zeta_{|D|}$.

For a flat torus $\mathbb{R}^n/\mathbb{Z}^n$:

$$N_\Delta(x) = c \cdot x^{n/2} + O(x^{(n-1)/2-\gamma}) \text{ for some } \gamma > 0.$$

The case of the free group

$G = \mathbb{F}_p$ with $p \geq 2$. ℓ is the length function, L is the multiplication operator by ℓ in the Hilbert space $H = \ell^2(G)$.

This is a case where F_L is not asymptotically continuous
$\rightarrow$ the previous results do not apply

However, we have explicit formulas for multiplicities and cumulated multiplicities : $sp(L) = \mathbb{N}$

$$m_k = \sharp\{g \in G\,,\ \ell(g) = k\} = 2p(2p-1)^{k-1}$$

$$M_k = \sharp\{g \in G\,,\ \ell(g) \leq k\} = \frac{2p}{2p-2}\big((2p-1)^k - 1\big)$$

Results

1/ $Tr_\omega(\rho(L)) = \dfrac{2p-2}{(2p-1)\mathrm{Log}(2p-1)}$

2/ The function ζ_L extends meromorphically to the half plane $Re(s) > 0$ with unique pole at $s = 1$ and residue $\dfrac{2p-2}{(2p-1)\mathrm{Log}(2p-1)}$.

Third part

Volume forms

The volume form associated with L

Definition

Let us fix a Dixmier ultrafilter ω. The volume form will be

$$\Omega_L \ : \ \mathcal{B}(H) \ni T \to Tr_\omega(T\rho(L))\,.$$

Ω_L positive linear form on $\mathcal{B}(H)$, $||\Omega_L|| \leq 1$

hence continuous for the uniform norm.

Obviously vanishing for T with finite rank, hence for T compact.

$\rightarrow$ it defines a positive linear form on the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$.

If F_L is asymptotically continuous, $F_L \sim \varphi$ at $+\infty$

then Ω_L is a state and

$$\Omega_L(T) = Tr_\omega(T\,\varphi(L)^{-1})^{\cdot}$$

In order to go further, we need to choose accurately the ultrafilter ω.

A. Carey, J. Phillips, F. Sukochev's ultrafilters

(Spectral flow and Dixmier traces, *Adv. in Math.* 173 (2003), 68-113)

There exist ultrafilters $\omega \in L^\infty(\mathbb{R}_+)$satisfying those requirements :

1/ $ess \liminf_{t\to+\infty} f(t) \leq \omega(f) \leq \limsup_{t\to+\infty} f(t) \quad f \in L^\infty(\mathbb{R}_+)$

Then, with the notation $\omega(f) = \omega - \lim_{t\to+\infty} f(t)$

2/ $\omega - \lim_{t\to+\infty} f(st) = \omega - \lim_{t\to+\infty} f(t) \quad f \in L^\infty(\mathbb{R}_+) \quad s \in \mathbb{R}_+^*$

3/ $\omega - \lim_{t\to+\infty} f(t^s) = \omega - \lim_{t\to+\infty} f(t) \quad f \in L^\infty(\mathbb{R}_+) \quad s \in \mathbb{R}_+^*$

4/ $\omega - lim_{t\to+\infty} \dfrac{1}{\mathrm{Log}(t)} \displaystyle\int_0^t f(s) \frac{ds}{s} = \omega - \lim_{t\to+\infty} f(t) \quad f \in L^\infty(\mathbb{R}_+)$

To such ω will be associated the Connes-Dixmier ultrafilter on $\mathbb{N}$

$$\omega - lim_{n\to\infty} f(n) = \omega - \lim_{t\to+\infty} f([t]) \quad f \in \ell^\infty(\mathbb{N})\,.$$

With such ultrafilter, for $\rho \in \mathcal{L}^{1,\infty}(H)_+$, $Tr_\omega(\rho)$ can be obtained as an ultra-limit as $s \downarrow 1+$ of the $(s-1)Tr(\rho^s)$ the following sense :

Theorem (CPS)

1/ $Tr_\omega(\rho) = \widetilde{\omega} - \lim_{r\to\infty} \frac{1}{r} Tr(\rho^{1+\frac{1}{r}})$

con $\widetilde{\omega} - \lim_{r\to\infty} f(r) = \omega - \lim_{r\to\infty} \mathrm{Log}(f(r))$.

2/ For $T \in \mathcal{B}(H)$, $Tr_\omega(T\,\rho) = \widetilde{\omega} - \lim_{r\to\infty} \frac{1}{r} Tr\,(T\,\rho^{1+\frac{1}{r}})$.

Application to discrete groups

G is a discrete group
λ is its left regular representation in $\ell^2(G)$
$C^*_{red}(G) = C^*(\lambda(G))$ its reduced C^*-algebra.

Let $\ell : G \to \mathbb{R}_+$ be any proper function and L the multiplication operator by ℓ in $\ell^2(L)$.
Computation :

For $g, g' \in G$, $g \neq 1_G$, and for $s > 1$, we compute
$$< \delta_{g'}, \lambda(g)\rho(L)^s\delta_{g'} >= F_L(g')^{-s} < \delta_{g'}, \delta_{gg'} >= 0$$
from which we deduce first $Tr(\lambda(g)\rho(L)^s) = 0$, then $Tr_\omega\big(\lambda(g)\rho(L)\big) = 0$.

Conclusion :

$$\text{For } a \in C^*_{red}(G)\,,\ \Omega_L(a) = Tr_\omega\big(\rho(L)\big)\,\tau(a)$$
with τ the canonical trace on $C^*_{red}(G)$.

Another example

Let $A \subset \mathcal{B}(H)$ a C^*-algebra which is an extension of the compact operators

$$0 \longrightarrow \mathcal{K}(H) \xrightarrow{j} A \xrightarrow{\sigma} C(X) \longrightarrow 0.$$

For any L on H, Ω_L vanishes on $\mathcal{K}$ hence factorizes through σ.

Conclusion :

There exists a measure μ su X such that

$$\forall a \in A, \Omega_L(a) = \int_X \sigma(a)\, d\mu \,.$$

In particular, the restriction to A of Ω_L is a trace.

This applies to the Toeplitz algebra in $\mathcal{B}(\ell^2(\mathbb{N}))$, where L is the operator of multiplication by n.

In the sequel, we present criteria of more analytical nature for obtaining a trace.

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A first criterion for Ω_L to be a trace on A

Let us define the Lipschitz algebra of L and its closure :

$$\mathcal{A} = \{a \in \mathcal{B}(H)\,,\ [L,a]\,\text{bounded}\}\ \text{and}\ A = \overline{\mathcal{A}}\,.$$

Assumption 1

$F_L(x) \sim \varphi(x), x \to +\infty$ with $\varphi \in W^{1,1}_{loc}(\mathbb{R}_+)$ and

$$ess\,\lim_{x\to+\infty} \varphi'(x)/\varphi(x) = 0\,.$$

Conclusions

1/ A lies in the centralizer of Ω_L .
2/ The restriction of Ω_L to A is a trace.
3/ Ω_L is an hypertrace on $\mathcal{B}(H)$:

$$\Omega_L(ba) = \Omega_L(ab)\,,\ a \in A\,,\ b \in \mathcal{B}(H)\,.$$

Observation 1 : we have as well $\Omega_L(T) = Tr_\omega(T\varphi(L)^{-1})$.

Observation 2 : $\lim_{x\to\infty} e^{-\beta x}\varphi(x) = 0$ for any $\beta > 0$
(φ and F_L have subexponential growth).

More criteria

Conclusions

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3/ Ω_L is an hypertrace on $\mathcal{B}(H)$:

$$\Omega_L(ba) = \Omega_L(ab)\,,\ a \in A\,,\ b \in \mathcal{B}(H)\,.$$

Assumption 2

$$\text{If } \lim_{k\to\infty} \frac{m_k}{M_k} = 0 \quad \text{and} \quad \frac{m_k}{M_k} = o(\widetilde{\lambda}_{k+1} - \widetilde{\lambda}_k),$$

then the conclusions hold true.

In particular

$$\text{If } \lim_{k\to\infty} \frac{m_k}{M_k} = 0 \quad \text{and } \liminf_{k\to\infty} \widetilde{\lambda}_{k+1} - \widetilde{\lambda}_k > 0$$

then the conclusions hold true.

A concluding remark

If $sp(L) \subset \mathbb{N}$, then the conclusions hold true as soon as F_L is asymptotically continuous.

For example :

a) ℓ the length function on a finitely generated discrete group :

$$\Omega_L \text{ is an hypertrace as soon as } \lim_k \frac{|B_{k+1}|}{|B_k|} = 1\,.$$

(this implies subexponential growth, hence amenability.)

b) D.V.V.'s example :

$$\Omega_D \text{ is an hypertrace on } A \text{ as soon as } \lim_k \frac{rk\,P_{k+1}}{rk\,P_k} = 1\,.$$