Popa's averaging property for automorphisms on C^*-algebras

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Probabilistic Operator Algebra Seminar
Berkeley, March 13, 2023

Outline

1. C^*-irreducible inclusions
2. An averaging technique of Popa
3. Inclusions of C^*-algebras arising from inclusions of groups

Given a unital inclusion $B \subseteq A$ of (unital, simple) C^*-algebras, what can you say about its intermediate C^*-algebras?

Definition: $B \subseteq A$ is C^**-irreducible* if all intermediate C^*-algebras $B \subseteq D \subseteq A$ are simple.

von Neumann analogy: For an inclusion $\mathcal{N} \subseteq \mathcal{M}$ of vNalg TFAE:

1. $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}$,
2. all intermediate von Neumann algs $\mathcal{N} \subseteq \mathcal{P} \subseteq \mathcal{M}$ are factors,
3. $\bigvee_{u \in \mathcal{U}(\mathcal{N})} upu^* = 1$, for all non-zero projections $p \in \mathcal{M}$.

Theorem: A unital inclusion $B \subseteq A$ of C^*-algebras is C^*-irreducible (all intermediate C^*-algs are simple) iff each non-zero $a \in A^+$ is full rel. to B.

Def: a is full relatively to B if $\exists u_1, \dots, u_m \in \mathcal{U}(B)$ st $\sum_{j=1}^m u_j^* a u_j \geq 1_A$.

Fact: $B \subseteq A$ is C^*-irreducible $\Rightarrow$ $B \subseteq A$ is irreducible (i.e., $B' \cap A = \mathbb{C}$).

- $B' \cap A = \mathbb{C}$ and A, B simple unital $\not\Rightarrow$ $B \subseteq A$ is C^*-irreducible.

Definition: Given inclusion $B \subseteq A$ of C^*-algs and cond. expect. $E \colon A \to B$, set

$$\mathrm{Ind}(E) = \lambda^{-1}, \qquad \lambda = \sup\{t \geq 0 \mid \forall a \in A^+ : E(a) \geq ta\}.$$

Theorem [Izumi, 2002]: Given $B \subseteq A$ and cond. expect. $E \colon A \to B$ with $\mathrm{Ind}(E) < \infty$.

- If A (or B) is simple, then B (or A) is a finite direct sum of simple C^*-algebras.
- In particular, if $A \cap B' = \mathbb{C}$, then A is simple iff B is simple.

Corollary: Given $B \subseteq A$ simple with cond. expect. $E \colon A \to B$ st $\mathrm{Ind}(E) < \infty$. Then:

$$B \subseteq A \text{ is } C^*\text{-irreducible} \iff A \cap B' = \mathbb{C}.$$

Definition: Given unital inclusion $B \subseteq A$ of C^*-algs, and $a \in A$. Set

$$C_B(a) = \overline{\mathrm{conv}\{u^*au : u \in \mathcal{U}(B)\}}.$$

- A has *Dixmier property* if $C_A(a) \cap \mathbb{C}1_A \neq \emptyset \ \forall a \in A$
- $B \subseteq A$ has the *relative Dixmier property* if $C_B(a) \cap \mathbb{C}1_A \neq \emptyset \ \forall a \in A$

Theorem [Popa]: $B \subseteq A$ has relative Dixmier property if

- B has Dixmier property,
- $B \subseteq A$ has finite Jones index wrt some cond. expect. $E\colon A \to B$,
- $\pi_\varphi(B)' \cap \pi_\varphi(A)'' = \mathbb{C}$, for some state φ on A.

- For $a \in A^+$: $C_B(a) \cap \mathbb{C}^\times \cdot 1_A \neq \emptyset \implies a$ is full rel. to B.

$B \subseteq A$ has rel. Dixmier property and A has a faithful tracial state, then $B \subseteq A$ is C^*-irreducible. (In particular, A and B must be simple.)

Theorem [Popa]: Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of separable II_1-factors. TFAE:

(i) $\mathcal{N} \subseteq \mathcal{M}$ is C^*-irreducible,

(ii) $\mathcal{N} \subseteq \mathcal{M}$ has the relative Dixmier property,

(iii) $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}$ and $[\mathcal{M} : \mathcal{N}] < \infty$.

(ii) $\iff$ (iii) is the main result of a paper of Popa.

(ii) $\implies$ (i) already noted.

(i) $\implies \mathcal{N}' \cap \mathcal{M} = \mathbb{C}$ also already noted.

(i) $\implies [\mathcal{M} : \mathcal{N}] < \infty$ follows from results of Popa, resp., F. Pop.

$$\begin{aligned}
\mathcal{N} \subseteq \mathcal{M} \ C^*\text{-irr} \quad &\iff \quad \forall 0 \neq p \in \mathcal{M}\ \exists u_1, \dots, u_n \in \mathcal{U}(\mathcal{N}) : \sum_{j=1}^{n} u_j p u_j^* \geq 1 \\
&\implies \quad \forall\, 0 \neq p \in \mathcal{M}\ \exists u_1, \dots, u_n \in \mathcal{U}(\mathcal{N}) : \vee_j\, u_j p u_j^* = 1 \\
&\implies \quad \forall 0 \neq p \in \mathcal{M} : \vee_{u \in \mathcal{U}(\mathcal{N})}\, u p u^* = 1 \\
&\iff \quad \mathcal{N}' \cap \mathcal{M} = \mathbb{C}
\end{aligned}$$

Outline

Definition: An automorphism α on a unital C^*-alg A has the *averaging property* (AveP) if

$$0 \in \overline{\mathrm{conv}\{v\alpha(v)^* : v \in \mathcal{U}(A)\}},$$

and the *strong averaging property* (SAveP) if

$$\forall a \in A : \quad 0 \in \overline{\mathrm{conv}\{va\alpha(v)^* : v \in \mathcal{U}(A)\}}.$$

- If $\alpha_1, \dots, \alpha_n \in \mathrm{Aut}(A)$ have (SAvP) and $a_1, \dots, a_n \in A$, then $\forall \varepsilon > 0$ $\exists v_1, \dots, v_m \in \mathcal{U}(A)$ st

$$\Big\| \frac{1}{m} \sum_{j=1}^{m} v_j a_i \alpha_i(v_j)^* \Big\| < \varepsilon, \quad i = 1, \dots, n.$$

$\alpha \in \mathrm{Aut}(A) \rightsquigarrow \bar{\alpha} \in \mathrm{Aut}(A^{**})$, $A^{**} = A^{**}_{\mathrm{fin}} \oplus A^{**}_{\mathrm{inf}}$, $\bar{\alpha} = \bar{\alpha}_{\mathrm{fin}} \oplus \bar{\alpha}_{\mathrm{inf}}$.

$\bar{\alpha}$ inner $\Rightarrow$ α inner, when A simple unital (Kishimoto)

α outer $\not\Rightarrow$ $\bar{\alpha}$ properly outer, not even when A simple.

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The following result appeared in a proof of Popa showing when an inclusion $A \subseteq A \rtimes_r \Gamma$ has the relative Dixmier property:

Theorem (Popa): Let A be unital simple C^*-alg w/ Dixmier property, let $a_1, \dots, a_n \in A$, $\alpha_1, \dots, \alpha_n \in \mathrm{Aut}(A)$, and assume all $\bar{\alpha}_i$ properly outer.
Then $\forall \varepsilon > 0 \ \exists v_1, \dots, v_m \in \mathcal{U}(A)$ st

$$\Big\| \frac{1}{m} \sum_{j=1}^{m} v_j a_i \alpha_i(v_j)^* \Big\| < \varepsilon, \quad i = 1, \dots, n.$$

Popa's thm—rephrased: If A simple w/ Dixmier property and $\alpha \in \mathrm{Aut}(A)$, then $\bar{\alpha}$ properly outer $\implies$ α has (SAveP).

Def: $\alpha \in \mathrm{Aut}(A)$ is *residually (properly) outer* if $\dot{\alpha} \in \mathrm{Aut}(A/I)$ is (properly) outer $\forall I \lhd_\alpha A$.

Theorem: Let A be a unital separable C^*-alg and $\alpha \in \mathrm{Aut}(A)$.

(i) α has (SAveP),

(ii) $\forall u \in \mathcal{U}(A) : \mathrm{Ad}_u \circ \alpha$ has (AveP),

(iii) α is residually outer and $\bar{\alpha}_{\mathrm{fin}}$ is properly outer.

(iv) α is residually properly outer and $\bar{\alpha}_{\mathrm{fin}}$ is properly outer.

Then (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

If $\mathrm{sr}(A) = 1$, then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Corollary: For an automorphism α on a unital *commutative* C^*-alg A,

$$\alpha \text{ has (AveP)} \iff \alpha \text{ has (SAveP)} \iff \alpha \text{ is free.}$$

Corollary: For an automorphism α on a unital, sep. *simple* C^*-alg A:

- If A has a trace, then α has (SAveP) $\iff$ $\bar{\alpha}_{\mathrm{fin}}$ is properly outer,
- if A has no trace, then α has (SAveP) $\iff$ α is outer.

On (i) $\iff$ (ii): For $u, v \in \mathcal{U}(A)$: $v(\mathrm{Ad}_u \circ \alpha)(v)^* = vu\alpha(v)^* u^*$. Hence

$$0 \in \overline{\mathrm{conv}\{vu\alpha(v)^* : v \in \mathcal{U}(A)\}} \Leftrightarrow 0 \in \overline{\mathrm{conv}\{v(\mathrm{Ad}_u \circ \alpha)(v)^* : v \in \mathcal{U}(A)\}}$$

This shows (i) $\Rightarrow$ (ii).

Conversely, suppose $\mathrm{Ad}_u \circ \alpha$ has (AveP) $\forall u \in \mathcal{U}(A)$, and $\mathrm{sr}(A) = 1$. Let $a \in A$. Since

$$\mathrm{conv}\{vb\alpha(v)^* : v \in \mathcal{U}(A)\} \subseteq \mathrm{conv}\{va\alpha(v)^* : v \in \mathcal{U}(A)\} := C,$$

for every $b \in C$, it suffices to show that $\exists b \in C$ st $\|b\| \leq \frac{3}{4}\|a\|$.

Since $\mathrm{sr}(A) = 1$, $\exists u_1, u_2, u_3 \in \mathcal{U}(A)$ st

$$a = \frac{1}{3}\|a\|(u_1 + u_2 + u_3) = \frac{1}{3}\|a\|u_1 + a_0, \qquad \|a_0\| = \frac{2}{3}\|a\|.$$

By the argument above, we get $0 \in \overline{\mathrm{conv}\{vu_1\alpha(v)^* : v \in \mathcal{U}(A)\}}$, which implies that

$$\inf\{\|b\| : b \in C\} \leq \|a_0\| = \frac{2}{3}\|a\| < \frac{3}{4}\|a\|,$$

as wanted.

Example: of an outer automorphism on a UHF algebra without (SAveP).

Set $A = \bigotimes_{n=1}^{\infty} M_{k_n}(\mathbb{C})$ with unique tracial state τ, where $k_n \geq 2$.
Set $u_n = \mathrm{diag}(1, 1, \ldots, 1, -1) \in \mathcal{U}(M_{k_n}(\mathbb{C}))$, and set

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}_{u_n} \in \mathrm{Aut}(A).$$

Then α is outer on A, while α extends to an inner automorphism $\bar{\alpha}$ on $M := \pi_\tau(A)''$ if $\sum_{n=1}^{\infty} \|1_{k_n} - u_n\|_2 < \infty$. Assume this is the case.
Let $v \in \mathcal{U}(M)$ implement $\bar{\alpha}$. Choose $w \in \mathcal{U}(A)$ st $\bar{\tau}(wv) \neq 0$. Since $uw\alpha(u)^* = uwvu^*v^*$, $\forall u \in \mathcal{U}(A)$, we get

$$C := \mathrm{conv}\{uw\alpha(u)^* : u \in \mathcal{U}(A)\} = \mathrm{conv}\{uwvu^* : u \in \mathcal{U}(A)\}v^*$$

Now, $\bar{\tau}(x) = \bar{\tau}(wv) \neq 0$, $\forall x \in \mathrm{conv}\{uwvu^* : u \in \mathcal{U}(A)\}$, which shows that $0 \notin \bar{C}$. Hence α fails (SAveP).

▶ If $\Gamma \curvearrowright A$ (SAveP) action and $x = \sum_{t \in \Gamma} a_t u_t \in A \rtimes_r \Gamma$, then $\forall t \in \Gamma$, $t \neq e$, $\exists v_1, \dots, v_m \in \mathcal{U}(A)$ st

$$\frac{1}{m} \sum_{j=1}^{m} v_j a_t u_t v_j^* = \Big(\frac{1}{m} \sum_{j=1}^{m} v_j a_t \alpha_t(v_j)^* \Big) u_t,$$

can be made arbitrarily small. Using this observation, one can prove

Lemma: Let $\Gamma \curvearrowright A$ be a (SAveP), let $x \in A \rtimes_r \Gamma$, and let $t \in \Gamma$.

(i) $E(x) = 1_A \implies 1_A \in C_A(x)$,

(ii) $E_t(x) = 1_A \implies u_t \in C_A^{\alpha_{t^{-1}}}(x) \subseteq C^*(A, x)$,

(iii) $\mathrm{dist}(C_A(x), A) = 0$.

- $E \colon A \rtimes_r \Gamma \to A$ is the standard cond. exp.
- $E_t(x) = E(x u_t^*) \in A$ is the "t-th Fourier coefficient" of $x \in A \rtimes_r \Gamma$.
- $C_A(x) = \overline{\mathrm{conv}\{u x u^* : u \in \mathcal{U}(A)\}}$.
- $C_A^{\alpha}(x) = \overline{\mathrm{conv}\{u x \alpha(u)^* : u \in \mathcal{U}(A)\}}$.

Given $\Gamma \curvearrowright A$ with A unital. Then $\exists$ trace on $A \rtimes_r \Gamma$ iff $\exists$ invariant trace τ on A (given by $\tau \circ E$). When is the extension of τ to $A \rtimes_r \Gamma$ unique?

Theorem (cf. Bedos, Thomsen, Ursu): Let $\Gamma \curvearrowright A$ be a (SAveP) action. Then each Γ-invariant trace τ on A extends *uniquely* to a trace on $A \rtimes_r \Gamma$. In fact, $\exists!$ state ρ on $A \rtimes_r \Gamma$ that extends τ and satisfies $\rho(uxu^*) = \rho(x)$ for all $A \rtimes_r \Gamma$ and all $u \in \mathcal{U}(A)$.

Proof: $\forall x \in A \rtimes_r \Gamma :\ \mathrm{dist}(A, \overline{\mathrm{conv}\{uxu^* : u \in \mathcal{U}(A)\}}) = 0$.

Theorem (cf. Popa): Let $\Gamma \curvearrowright A$ be an action on a unital C^*-alg A. Then $A \subseteq A \rtimes_r \Gamma$ has the relative Dixmier property if and only if A has the Dixmier property and the action is (SAveP).

Proof: "If:" As in the proof above! "Only if:" For $t \neq e$ and $a \in A$, set

$$M := \mathrm{conv}\{v a u_t v^* : v \in \mathcal{U}(A)\} = \mathrm{conv}\{v a \alpha_t(v)^* : v \in \mathcal{U}(A)\} u_t$$

Then $\overline{M} \subseteq \mathbb{C} u_t$, $\overline{M} \cap \mathbb{C} \neq \emptyset$, and $\mathbb{C} u_t \cap \mathbb{C} = \{0\}$. Hence $0 \in \overline{M}$.

Theorem (cf. Izumi, Cameron-Smith, Bedos–Omland): Let $\Gamma \curvearrowright A$ be a (SAveP) action, let $\Lambda \leq \Gamma$, and suppose $\Lambda \curvearrowright A$ is minimal. Then

(i) $A \rtimes_r \Lambda \subseteq A \rtimes_r \Gamma$ is C^*-irreducible,

(ii) If $\Lambda \lhd \Gamma$, then

$$\{A \rtimes_r \Lambda \subseteq D \subseteq A \rtimes_r \Gamma\} \longleftrightarrow \{\Lambda \leq \Upsilon \leq \Gamma\}, \quad \Upsilon \mapsto A \rtimes_r \Upsilon.$$

Proof: Let $A \rtimes_r \Lambda \subseteq D \subseteq A \rtimes_r \Gamma$, let $J \lhd D$, let $t \in \Gamma$, and set

$$E_t(J) = \{E_t(x) : x \in J\} \subseteq A, \qquad [\text{Recall: } E_t(x) = E(xu_t^*)]$$

- $E_t(J) \lhd A$.
- $E_t(J)$ is Λ invariant if $[t, \Lambda] \subseteq \Lambda$, eg., if $t = e$ or if $\Lambda \lhd \Gamma$.

(i). Let $J \lhd D$ be as above. Then, with $t = e$ and $E_t = E$,

$$J \neq 0 \;\Rightarrow\; E(J) = A \;\Rightarrow\; 1_A \in E(J) \overset{(*)}{\Rightarrow} 1_A \in C_A(J) \subseteq J \;\Rightarrow\; J = D$$

$(*)$ by previous lemma (using the action is (SAveP)).

Theorem (cf. Izumi, Cameron-Smith, Bedos–Omland): Let $\Gamma \curvearrowright A$ be a (SAveP) action, let $\Lambda \leq \Gamma$, and suppose $\Lambda \curvearrowright A$ is minimal. Then

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$$\{A \rtimes_r \Lambda \subseteq D \subseteq A \rtimes_r \Gamma\} \longleftrightarrow \{\Lambda \leq \Upsilon \leq \Gamma\}, \quad \Upsilon \mapsto A \rtimes_r \Upsilon.$$

Proof: (ii). Let $A \rtimes_r \Lambda \subseteq D \subseteq A \rtimes_r \Gamma$, let $t \in \Gamma$, and set

$$E_t(D) = \{E_t(x) : x \in D\} \subseteq A.$$

- $E_t(D) \lhd A$ and $E_t(D)$ is Λ invariant, since $\Lambda \lhd \Gamma$. Hence

$$E_t(D) \neq 0 \;\Rightarrow\; E_t(D) = A \;\Rightarrow\; 1_A \in E_t(D) \overset{(*)}{\Rightarrow} u_t \in C_A^{\alpha_{t^{-1}}}(D) \subseteq D$$

Set $\Upsilon = \{t \in \Gamma : u_t \in D\} = \{t \in \Gamma : E_t(D) \neq 0\}$. Then $\Lambda \leq \Upsilon \leq \Gamma$.

Clearly, $A \rtimes_r \Upsilon \subseteq D$. If $x \in D$, then $\mathrm{supp}(x) = \{t \in \Gamma : E_t(x) \neq 0\} \subseteq \Upsilon$, whence $x \in A \rtimes_r \Upsilon$. This shows $D \subseteq A \rtimes_r \Upsilon$.

Outline

We now consider inclusions of C^*-algs (and von Neumann algs) arising from inclusions $\Lambda \subseteq \Gamma$ of groups.

Definition: Γ is icc rel. to Λ iff $\{tst^{-1} : t \in \Lambda\}$ is infinite $\forall s \in \Gamma \setminus \{e\}$.

Proposition: Given groups $\Lambda \subseteq \Gamma$. Then
$\mathcal{L}(\Lambda) \subseteq \mathcal{L}(\Gamma)$ is C^*-irreducibe $\iff$ Γ is icc rel. to Λ and $[\Gamma : \Lambda] < \infty$.

Proof:

- $\mathcal{L}(\Lambda)' \cap \mathcal{L}(\Gamma) = \mathbb{C} \iff \Gamma$ is icc relatively to Λ.
- $[\mathcal{L}(\Gamma) : \mathcal{L}(\Lambda)] = [\Gamma : \Lambda] < \infty$.

We proceed to consider when $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ is C^*-irreducible. A few quick facts (the second follows from Popa's theorem):

- $C^*_\lambda(\Lambda)' \cap C^*_\lambda(\Gamma) = \mathbb{C} \iff \Gamma$ is icc rel. to Λ.
- $|\Gamma : \Lambda| < \infty$: $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ C^*-irr. $\iff$ Γ icc rel. to Λ, Γ C^*-simple.
- $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ C^*-irreducible $\not\Rightarrow [\Gamma : \Lambda] < \infty$.

Theorem: Let $\Lambda \subseteq \Gamma$ be groups.

(i) $\exists\, \Gamma \curvearrowright X$ top. free bdry action st $\forall \mu \in \mathrm{Prob}(X)\ \exists\, \delta_x \in \overline{\Lambda.\mu}$ for which Γ acts freely on x,

(ii) $\tau_0 \in \overline{\{s.\varphi : s \in \Lambda\}}^{\mathrm{weak}^*}$, for all states φ on $C^*_\lambda(\Gamma)$,

(iii) $\tau_0 \in \overline{\mathrm{conv}\{s.\varphi : s \in \Lambda\}}^{\mathrm{weak}^*}$, for all states φ on $C^*_\lambda(\Gamma)$,

(iv) The relative Powers' averaging procedure holds: $\forall s_1, \dots, s_n \in \Gamma \setminus \{e\}$ $\forall \varepsilon > 0\ \exists t_1, \dots, t_m \in \Lambda$ st $\|\frac{1}{m}\sum_{k=1}^{m} \lambda(t_k s_j t_k^{-1})\| \leq \varepsilon$, for $j = 1, \dots, n$.

(v) $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ has the relative Dixmier property,

(vi) $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ is C^*-irreducible.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (vi), & (vi) $\Rightarrow$ (v) if $[\Gamma : \Lambda] < \infty$.

- Condition (iv) is termed $\Lambda \subseteq \Gamma$ is *plump* by Amrutam–Ursu.

Example: $C^*_\lambda(\mathbb{F}_n) \subseteq C^*_\lambda(\mathbb{F}_m)$ is C^*-irreducible when $n < m$.

Scarparo: This inclusion also satisfies (i), with $X = \partial\mathbb{F}_m$ Gromov boundary, hence (i)–(vi) hold.

(i) $\exists\, \Gamma \curvearrowright X$ top. free bdry action st $\forall \mu \in \mathrm{Prob}(X)\; \exists\, \delta_x \in \overline{\Lambda.\mu}$ for which Γ acts freely on x,

(ii) $\tau_0 \in \overline{\{s.\varphi : s \in \Lambda\}}^{\text{weak}^*}$, for all states φ on $C^*_\lambda(\Gamma)$,

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(v) $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ has the relative Dixmier property,

(vi) $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ is C^*-irreducible.

Ursu: For Λ is **normal** in Γ: (iv) $\implies$ $\Gamma \curvearrowright \partial_F \Lambda$ is free $\implies$ (i). Hence (i)–(v) are equivalent.

Bedos–Omland/Li–Scarparo: For Λ is **commensurated** in Γ: (i)–(vi) are equivalent, and also equiv. to Γ icc relatively to Λ.

Bedos–Omland: $\exists$ C^*-simple groups $\Lambda \subseteq \Gamma$ with Γ icc rel. to Λ st $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ not C^*-irreducible.