SYK Model and q-Gaussian Distribution

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joint work with Miguel Pluma
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"Symbolic Tools in Mathematics and Their Application"

Section 1

Black Holes, Quantum Chaos, and Random Matrices

SYK=Sachdev–Ye–Kitaev

- strongly interacting quantum many body systems
- French and Wong (1970), Bohigas and Flores (1971): two body random hamiltonian
- Sachdev and Ye (1993): Gapless spin-fluid ground state in a random quantum Heisenberg magnet
- Kitaev 2015: "A simple model of quantum holography"

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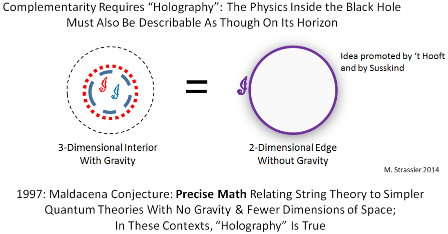

- gravity dual of SYK may be quantum Anti-de Sitter space in two bulk dimensions (AdS_2)

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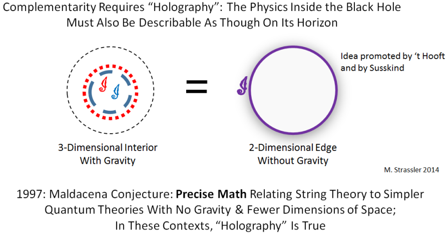

- gravity dual of SYK may be quantum Anti-de Sitter space in two bulk dimensions (AdS_2)
- black holes are chaotic systems, so their dual should also show chaos
- "quantum chaos" corresponds to "Random Matrix Theory correlations" (Bohigas–Giannoni-Schmit conjecture 1984)

Section 2

SYK Model

$$H_{n,q_n} := \frac{\sqrt{-1}^{\lfloor \frac{q_n}{2} \rfloor}}{\binom{n}{q_n}^{\frac{1}{2}}} \sum_{1 \le i_1 < \cdots < i_{q_n} \le n} J_{i_1,\ldots,i_{q_n}} \psi_{i_1} \cdots \psi_{i_{q_n}} \qquad \text{(SYK)}$$

where

- $n \in \mathbb{N}$ is an even number
- $q_n \in \mathbb{N}$ natural number
  - might depend on n
  - $q_n = 4$ in original model
  - we will consider $q_n \sim n^{1/2}$
- $\psi_1, \ldots, \psi_n$ are Majorana fermions, i.e., variables which fulfill the following relations

$$\psi_i \psi_j + \psi_j \psi_i = 2\delta_{ij}.$$

- where the random coefficients $J_{i_1,\ldots,i_{q_n}}$ are independent real random variables with moments of all orders and

$$\mathbb{E}[J_{i_1,\ldots,i_{q_n}}] = 0, \quad \mathbb{E}[J^2_{i_1,\ldots,i_{q_n}}] = 1.$$

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Random Matrix Realization

The Majoranas ψ_i can be realized in terms of

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

as r-fold tensor products (with $n = 2r$)

$$\begin{aligned} \psi_1 &= \sigma_1 \otimes 1 \otimes \cdots \otimes 1 & \psi_{r+1} &= \sigma_2 \otimes 1 \otimes \cdots \otimes 1 \\ \psi_2 &= \sigma_3 \otimes \sigma_1 \otimes \cdots \otimes 1 & \psi_{r+2} &= \sigma_3 \otimes \sigma_2 \otimes \cdots \otimes 1 \\ &\vdots & &\vdots \\ \psi_r &= \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_1 & \psi_{2r} &= \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_2 \end{aligned}$$

Thus H_{n,q_n} is a random matrix of size $N = 2^{n/2}$

Sparse Random Matrix Realization

The Majoranas ψ_i can be realized in terms of

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as r-fold tensor products (with $n = 2r$)

$$\begin{aligned} \psi_1 &= \sigma_1 \otimes 1 \otimes \cdots \otimes 1 & \psi_{r+1} &= \sigma_2 \otimes 1 \otimes \cdots \otimes 1 \\ \psi_2 &= \sigma_3 \otimes \sigma_1 \otimes \cdots \otimes 1 & \psi_{r+2} &= \sigma_3 \otimes \sigma_2 \otimes \cdots \otimes 1 \\ &\vdots & &\vdots \\ \psi_r &= \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_1 & \psi_{2r} &= \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_2 \end{aligned}$$

Thus H_{n,q_n} is a random matrix of size $N = 2^{n/2}$ and rank $\binom{n}{q_n}$, thus a sparse random matrix.

SYK Model

$$H_{n,q_n} := \frac{\sqrt{-1}^{\lfloor \frac{q_n}{2} \rfloor}}{\binom{n}{q_n}^{\frac{1}{2}}} \sum_{1 \leq i_1 < \cdots < i_{q_n} \leq n} J_{i_1,\dots,i_{q_n}} \psi_{i_1} \cdots \psi_{i_{q_n}}$$

where

- $n \in \mathbb{N}$ is an even number
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- $\psi_1, \dots, \psi_n$ are Majorana fermions, i.e., variables which fulfill the following relations

$$\psi_i \psi_j + \psi_j \psi_i = 2\delta_{ij}.$$

- where the random coefficients $J_{i_1,\dots,i_{q_n}}$ are i.i.d. classical random variables

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Dynamical Version of SYK Model

$$H_{n,q_n}(\mathbf{t}) := \frac{\sqrt{-1}^{\lfloor \frac{q_n}{2} \rfloor}}{\binom{n}{q_n}^{\frac{1}{2}}} \sum_{1 \le i_1 < \cdots < i_{q_n} \le n} J_{i_1,\dots,i_{q_n}}(\mathbf{t}) \psi_{i_1} \cdots \psi_{i_{q_n}}$$

where

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$$\psi_i \psi_j + \psi_j \psi_i = 2\delta_{ij}.$$

- where the random coefficients $J_{i_1,\dots,i_{q_n}}(\mathbf{t})$ are **independent classical Brownian motions**

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Section 3

Asymptotic Eigenvalue Distribution of SYK Model

Question

What is asymptotic eigenvalue distribution of

$$H_{n,q_n}(t) := \frac{\sqrt{-1}^{\lfloor \frac{q_n}{2} \rfloor}}{\binom{n}{q_n}^{\frac{1}{2}}} \sum_{1 \le i_1 < \cdots < i_{q_n} \le n} J_{i_1,\dots,i_{q_n}}(t)\psi_{i_1} \cdots \psi_{i_{q_n}}$$

for $n \to \infty$ with respect to $\mathbb{E} \circ \mathrm{tr}$?

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for $n \to \infty$ with respect to $\mathbb{E} \circ \mathrm{tr}$?

Answer: Convergence to the q-Gaussian distribution

The spectral density of $H_{n,4}$ is, for large n, close to the q-Gaussian distribution; actually in the double scaling limit

$$q_n \to \infty, \qquad \text{such that} \qquad \frac{q_n^2}{n} \to \lambda \in [0, \infty]$$

one has convergence of dist(H_{n,q_n}) to the q-distribution, with $q = e^{-2\lambda}$.

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- was observed in physics by Garcia-Garcia, Verbaarschot (2017), Cottler et al. (2017)
- had actually been proved before in the context of a similar model for quantum spin glasses by Erdös and Schröder (2014)
- was proved for SYK model by Feng, Tian, Wei (2019)

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Asymptotic distribution of the dynamical SYK model (Pluma + RS)

Consider the dynamical version of the SYK model:

$$H(t) := \frac{\sqrt{-1}^{\lfloor \frac{q_n}{2} \rfloor}}{\binom{n}{q_n}^{\frac{1}{2}}} \sum_{1 \le i_1 < \cdots < i_{q_n} \le n} J_{i_1,\ldots,i_{q_n}}(t) \psi_{i_1} \cdots \psi_{i_{q_n}},$$

where the $J_{i_1,\ldots,i_{q_n}}(t)$ (with $n \in \mathbb{N}$, $1 \le i_1 < \cdots < i_{q_n} \le n$) are independent classical Brownian motions. We assume the existence of the limit

$$\lim_{n\to\infty} \frac{q_n^2}{n} =: \lambda \in [0,\infty], \qquad \text{and put} \qquad q = \begin{cases} e^{-2\lambda} & \text{if } q_n \text{ are even} \\ -e^{-2\lambda} & \text{if } q_n \text{ are odd} \end{cases}$$

Then, the process $(H(t))_{t\ge 0}$ converges, for $n \to \infty$, to the q-Brownian motion $(S_q(t))_{t\ge 0}$, in the sense that we have for all $0 \le t_1, \ldots, t_k$ that

$$\lim_{n\to\infty} \mathbb{E}\left[\mathrm{tr}(H(t_1)\cdots H(t_k))\right] = \tau\left(S_q(t_1)\cdots S_q(t_k)\right).$$

Section 4

q-Brownian Motion

q-Fock space

Fix $q \in [-1, 1]$ and consider Hilbert space $\mathcal{H}$. The q-Fock space

$$\mathcal{F}_q(\mathcal{H}) = \overline{\bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}}^{\langle\cdot,\cdot\rangle_q} \qquad (\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega)$$

is completion of algebraic Fock space with respect to inner product

$$\langle h_1 \otimes \cdots \otimes h_n, g_1 \otimes \cdots \otimes g_m \rangle_q = \delta_{nm} \sum_{\sigma \in S_n} \prod_{r=1}^{n} \langle h_r, g_{\sigma(r)} \rangle q^{i(\sigma)},$$

- $i(\sigma) = \#\{(k,l) \mid 1 \leq k < l \leq n; \sigma(k) > \sigma(l)\}$ is number of inversions

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- inner product is positive definite, and has a kernel only for $q = 1$ and $q = -1$ (Bozejko + Speicher 1991)
- for $q = 1$ and $q = -1$ first divide out the kernel, thus leading to the symmetric and anti-symmetric Fock space, respectively.

Creation and annihilation operators, $a(h)$ and $a^*(h)$ for $h \in \mathcal{H}$

- $a^*(h)\Omega = h$ and $a^*(h)h_1 \otimes \cdots \otimes h_n = h \otimes h_1 \otimes \cdots \otimes h_n$
- its adjoint is given by $a(h)\Omega = 0$ and

$$a(h)h_1 \otimes \cdots \otimes h_n = \sum_{r=1}^{n} q^{r-1}\langle h, h_r\rangle h_1 \otimes \cdots \otimes h_{r-1} \otimes h_{r+1} \otimes \cdots \otimes h_n.$$

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- those operators satisfy the q-commutation relations

$$a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle \cdot 1 \qquad (f, g \in \mathcal{H}).$$

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$$a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle \cdot 1 \qquad (f, g \in \mathcal{H}).$$

- prominent special cases:
  - $q = 1$: CCR relations
  - $q = 0$: Cuntz relations
  - $q = -1$: CAR relations
- with the exception of the case $q = 1$, the operators $a^*(f)$ are bounded

q-Gaussian Distribution

- consider q-Gaussian operators

$$s_q(h) = a(h) + a^*(h) \qquad h \in \mathcal{H}_{\text{real}}$$

- consider vacuum expectation state

$$\tau(T) = \langle \Omega, T\Omega \rangle_q, \quad \text{for} \quad T \in \mathcal{B}(\mathcal{F}_q(\mathcal{H})).$$

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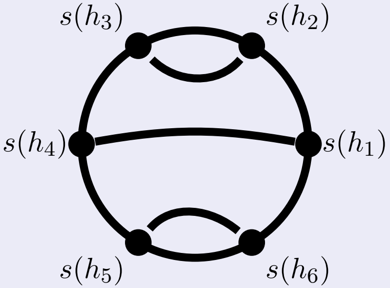
- multivariate q-Gaussian distribution is the non commutative distribution of a collection of q-Gaussians with respect to the vacuum expectation state τ
- is given by q-deformed version of the Wick/Isserlis formula

$$\tau\left(s_q(h_{i(1)}) \cdots s_q(h_{i(k)})\right) = \sum_{\pi \in \mathcal{P}_2(k)} q^{cr(\pi)} \prod_{(r,s) \in \pi} \langle h_{i(r)}, h_{i(s)} \rangle,$$

where $cr(\pi)$ denotes number of crossings of pairing π.

Contribution of Pairing to Moment $\tau[s(h_1)s(h_2)s(h_3)s(h_4)s(h_5)s(h_6)]$

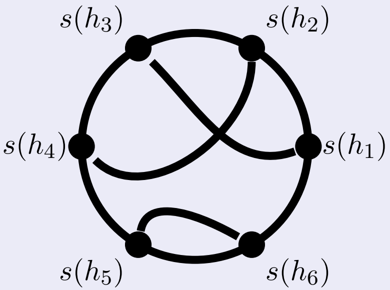
non-crossing



$\langle h_1, h_4\rangle \cdot \langle h_2, h_3\rangle \cdot \langle h_5, h_6\rangle$

Contribution of Pairing to Moment $\tau[s(h_1)s(h_2)s(h_3)s(h_4)s(h_5)s(h_6)]$

one crossing

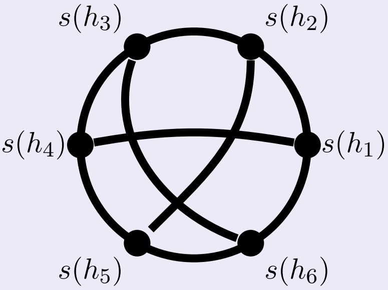


$$q \cdot \langle h_1, h_3 \rangle \cdot \langle h_2, h_4 \rangle \cdot \langle h_5, h_6 \rangle$$

Contribution of Pairing to Moment

$\tau[s(h_1)s(h_2)s(h_3)s(h_4)s(h_5)s(h_6)]$

three crossings



$$q^3 \cdot \langle h_1, h_4 \rangle \cdot \langle h_2, h_5 \rangle \cdot \langle h_3, h_6 \rangle$$

Section 5

Convergence of SYK to q-Brownian Motion

Asymptotic distribution of the dynamical SYK model (Pluma + RS)

Consider the dynamical version of the SYK model:

$$H(t) := \frac{\sqrt{-1}^{\lfloor \frac{q_n}{2} \rfloor}}{\binom{n}{q_n}^{\frac{1}{2}}} \sum_{1 \le i_1 < \cdots < i_{q_n} \le n} J_{i_1,\dots,i_{q_n}}(t) \psi_{i_1} \cdots \psi_{i_{q_n}},$$

We assume the existence of the limit

$$\lim_{n\to\infty} \frac{q_n^2}{n} =: \lambda \in [0,\infty], \qquad \text{and put} \qquad q = \begin{cases} e^{-2\lambda} & \text{if } q_n \text{ are even} \\ -e^{-2\lambda} & \text{if } q_n \text{ are odd} \end{cases}$$

Consider the q-Brownian motion

$$S_q(t) := s_q(1_{[0,t]}) \qquad \text{on} \qquad \mathcal{F}_q(L^2(\mathbb{R}_+)).$$

Then, for all $0 \le t_1, \dots, t_k$,

$$\lim_{n\to\infty} \mathbb{E}\left[\operatorname{tr}(H(t_1)\cdots H(t_k))\right] = \tau\left[S_q(t_1)\cdots S_q(t_k)\right].$$

Calculation of Moments of SYK

For $I = (1 \leq i_1 < i_2 < \cdots < i_{q_n} \leq n)$ we put

$$J_I := J_{i_1,\dots,i_{q_n}} \qquad \text{and} \qquad \Psi_I := \sqrt{-1}^{\lfloor \frac{q_n}{2} \rfloor} \cdot \psi_{i_1} \cdots \psi_{i_{q_n}}$$

and write

$$H = c_n \sum_I J_I \Psi_I \qquad \text{with} \qquad c_n := \frac{1}{\binom{n}{q_n}^{\frac{1}{2}}}$$

Then

$$\mathbb{E}[\mathrm{tr}(H^6)] = c_n^6 \sum_{I_1,\dots,I_6} \mathbb{E}[J_{I_1} J_{I_2} \cdots J_{I_6}] \cdot \mathrm{tr}[\Psi_{I_1} \Psi_{I_2} \cdots \Psi_{I_6}]$$

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Then

$$\mathbb{E}[\mathrm{tr}(H^6)] = c_n^6 \sum_{I_1,\dots,I_6} \mathbb{E}[J_{I_1} J_{I_2} \cdots J_{I_6}] \cdot \mathrm{tr}[\Psi_{I_1} \Psi_{I_2} \cdots \Psi_{I_6}]$$

$$= c_n^6 \sum_{\pi \in \mathcal{P}_2(6)} \sum_{\substack{I_1,\dots,I_6 \\ \ker(I_1,\dots,I_6)=\pi}} \mathrm{tr}[\Psi_{I_1} \Psi_{I_2} \cdots \Psi_{I_6}]$$

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In order to calculate $\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\cdots\Psi_{I_6}]$ note

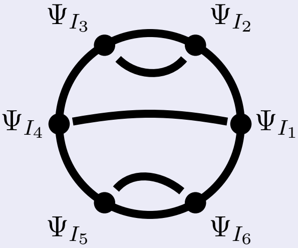
- $\Psi_I^2 = 1$,

$$\mathbb{E}[\mathrm{tr}(H^6)] = c_n^6 \sum_{\pi \in \mathcal{P}_2(6)} \sum_{\substack{I_1,\dots,I_6 \\ \ker(I_1,\dots,I_6)=\pi}} \mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\cdots\Psi_{I_6}]$$

In order to calculate $\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\cdots\Psi_{I_6}]$ note

- $\Psi_I^2 = 1$,

non-crossing



$$\mathrm{tr}[\Psi_{I_1} \underbrace{\Psi_{I_2}\Psi_{I_2}}_{1} \Psi_{I_1} \underbrace{\Psi_{I_5}\Psi_{I_5}}_{1}] = 1$$

sum over $(I_1, \dots, I_6)$ with this configuations gives contribution 1

$$\mathbb{E}[\mathrm{tr}(H^6)] = c_n^6 \sum_{\pi\in\mathcal{P}_2(6)} \sum_{\substack{I_1,\dots,I_6 \\ \ker(I_1,\dots,I_6)=\pi}} \mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\cdots\Psi_{I_6}]$$

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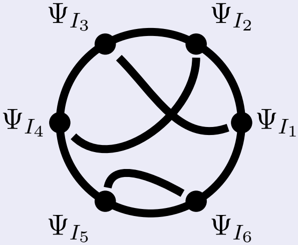

$$\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_1}\Psi_{I_2}\underbrace{\Psi_{I_5}\Psi_{I_5}}_{1}] = \mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_1}\Psi_{I_2}]$$

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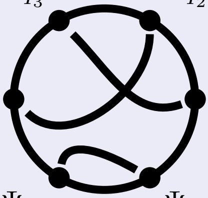
In order to calculate $\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\cdots\Psi_{I_6}]$ note

- $\Psi_I^2 = 1, \qquad \Psi_{I_1}\Psi_{I_2} = (-1)^{q_n + |I_1 \cap I_2|}\Psi_{I_2}\Psi_{I_1}$

one crossing



$$\begin{aligned}\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_1}\Psi_{I_2}\underbrace{\Psi_{I_5}\Psi_{I_5}}_{1}] &= \mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_1}\Psi_{I_2}] \\ &= \pm\,\mathrm{tr}[\Psi_{I_1}\Psi_{I_1}\Psi_{I_2}\Psi_{I_2}] \\ &= (-1)^{q_n + |I_1 \cap I_2|}\end{aligned}$$

$$\mathbb{E}[\mathrm{tr}(H^6)] = c_n^6 \sum_{\pi \in \mathcal{P}_2(6)} \sum_{\substack{I_1,\dots,I_6 \\ \ker(I_1,\dots,I_6)=\pi}} \mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\cdots\Psi_{I_6}]$$

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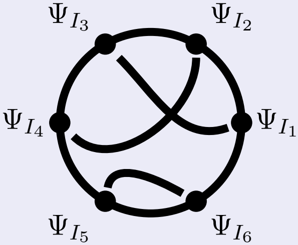
- $\Psi_I^2 = 1, \qquad \Psi_{I_1}\Psi_{I_2} = (-1)^{q_n + |I_1 \cap I_2|}\Psi_{I_2}\Psi_{I_1}$

one crossing



$$\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_1}\Psi_{I_2}\underbrace{\Psi_{I_5}\Psi_{I_5}}_{1}] = \mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_1}\Psi_{I_2}]$$
$$= \pm\,\mathrm{tr}[\Psi_{I_1}\Psi_{I_1}\Psi_{I_2}\Psi_{I_2}]$$
$$= (-1)^{q_n + |I_1 \cap I_2|}$$

- for $q_n = 4$, most I_1, I_2 have no indices in common: ± 1 independent of I_1, I_2
- sum over $(I_1, \dots, I_6)$ gives global factor $(-1)^{q_n}$

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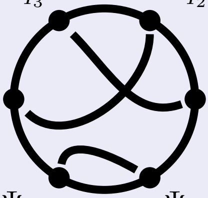

$$\begin{aligned}\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_1}\Psi_{I_2}\underbrace{\Psi_{I_5}\Psi_{I_5}}_{1}] &= \mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_1}\Psi_{I_2}]\\ &= \pm\,\mathrm{tr}[\Psi_{I_1}\Psi_{I_1}\Psi_{I_2}\Psi_{I_2}]\\ &= (-1)^{q_n+|I_1\cap I_2|}\end{aligned}$$

- for $q_n^2 \sim \lambda n$, $|I_1 \cap I_2|$ is asymptotically Poisson distributed
- sum over $(I_1,\dots,I_6)$ gives an averaged factor $q := (-1)^{q_n}e^{-2\lambda}$

$$\mathbb{E}[\mathrm{tr}(H^6)] = c_n^6 \sum_{\pi \in \mathcal{P}_2(6)} \sum_{\substack{I_1,\dots,I_6 \\ \ker(I_1,\dots,I_6)=\pi}} \mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\cdots\Psi_{I_6}]$$

In order to calculate $\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\cdots\Psi_{I_6}]$ note

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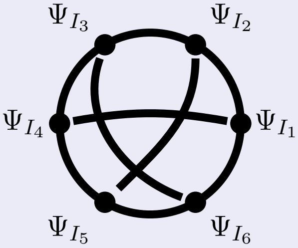
three crossings



$$\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_3}\Psi_{I_1}\Psi_{I_2}\Psi_{I_3}] = (-1)^{q_n + |I_1 \cap I_2|} \cdot (-1)^{q_n + |I_1 \cap I_3|} \cdot (-1)^{q_n + |I_2 \cap I_3|}$$

$$\mathbb{E}[\mathrm{tr}(H^6)] = c_n^6 \sum_{\pi \in \mathcal{P}_2(6)} \sum_{\substack{I_1,\dots,I_6 \\ \ker(I_1,\dots,I_6)=\pi}} \mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\cdots\Psi_{I_6}]$$

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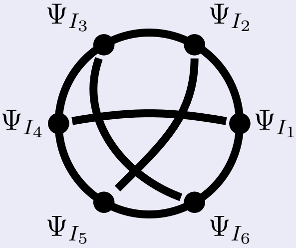

$$\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_3}\Psi_{I_1}\Psi_{I_2}\Psi_{I_3}]$$
$$= (-1)^{q_n+|I_1\cap I_2|} \cdot (-1)^{q_n+|I_1\cap I_3|} \cdot (-1)^{q_n+|I_2\cap I_3|}$$

- asymptotically the commutation factors corresponding to the three crossings are independent from each other
- sum over $(I_1,\dots,I_6)$ gives an averaged factor q^3

q-Gaussian as Stochastic Mixture of Commuting and Anti-Commuting Operators

Similar ideas for realizing the q-Gaussian distribution via a mixture of commuting/anti-commuting operators had appeared earlier

- Speicher: A non-commutative central limit theorem, 1992
- Sniady: Gaussian random matrix models for q-deformed Gaussian random variables, 2001
- Parisi: D-dimensional arrays of Josephson junctions, spin glasses and q-deformed harmonic oscillators, 1994

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Section 6

Fluctuations

higher correlations functions

Let k_m be m-th classical cumulant. Then consider m-th correlation function

$$k_m\left(\mathrm{tr}(H^{r_1}), \mathrm{tr}(H^{r_2}), \ldots, \mathrm{tr}(H^{r_m})\right)$$

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higher correlations functions

Let k_m be m-th classical cumulant. Then consider m-th correlation function

$$\lim_{n\to\infty}\binom{n}{q_n}^{m-1} k_m\left(\mathrm{tr}(H_{n,q_n}^{r_1}), \mathrm{tr}(H_{n,q_n}^{r_2}), \dots, \mathrm{tr}(H_{n,q_n}^{r_m})\right)$$

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higher correlations functions

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$k_2\left(\mathrm{tr}(H^6), \mathrm{tr}(H^4)\right)$

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$$k_2\left(\mathrm{tr}(H^6), \mathrm{tr}(H^4)\right)$$

$$= k_2\left(\sum_{I_1,\ldots,I_6} J_{I_1}\cdots J_{I_6}\,\mathrm{tr}[\Psi_{I_1}\cdots\Psi_{I_6}], \sum_{K_1,\ldots,K_4} J_{K_1}\cdots J_{K_4}\,\mathrm{tr}[\Psi_{K_1}\cdots\Psi_{K_4}]\right)$$

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higher correlations functions

Let k_m be m-th classical cumulant. Then consider m-th correlation function

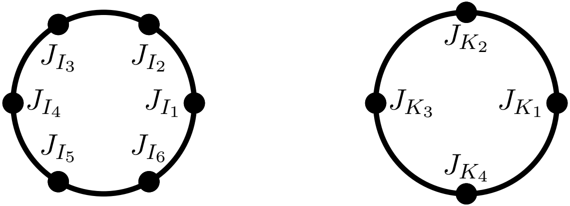
$$\lim_{n\to\infty} \binom{n}{q_n}^{m-1} k_m \left(\operatorname{tr}(H^{r_1}_{n,q_n}), \operatorname{tr}(H^{r_2}_{n,q_n}), \ldots, \operatorname{tr}(H^{r_m}_{n,q_n}) \right)$$

$$k_2 \left(\operatorname{tr}(H^6), \operatorname{tr}(H^4) \right)$$

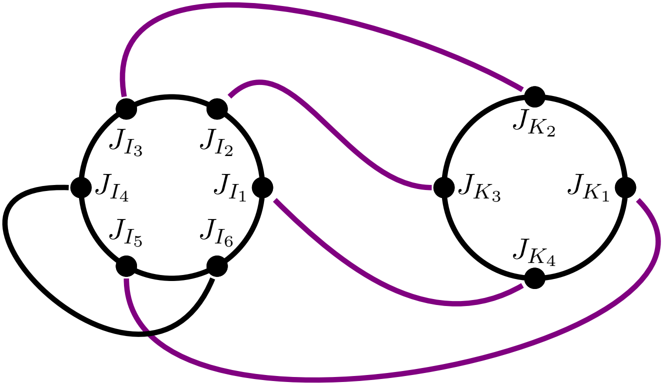
$$= k_2 \left(\sum_{I_1,\ldots,I_6} J_{I_1} \cdots J_{I_6} \operatorname{tr}[\Psi_{I_1} \cdots \Psi_{I_6}], \sum_{K_1,\ldots,K_4} J_{K_1} \cdots J_{K_4} \operatorname{tr}[\Psi_{K_1} \cdots \Psi_{K_4}] \right)$$

$$= \sum_{\substack{I_1,\ldots,I_6 \\ K_1,\ldots,K_4}} k_2(J_{I_1} \cdots J_{I_6}, J_{K_1} \cdots J_{K_4}) \cdot \operatorname{tr}[\Psi_{I_1} \cdots \Psi_{I_6}] \cdot \operatorname{tr}[\Psi_{K_1} \cdots \Psi_{K_4}]$$

Contribution to $k_2(\cdot,\cdot)\,\mathrm{tr}[\Psi_{I_1}\cdots\Psi_{I_6}]\,\mathrm{tr}[\Psi_{K_1}\cdots\Psi_{K_4}])$



Contribution to $k_2(\cdot,\cdot)\,\mathrm{tr}[\Psi_{I_1}\cdots\Psi_{I_6}]\,\mathrm{tr}[\Psi_{K_1}\cdots\Psi_{K_4}])$



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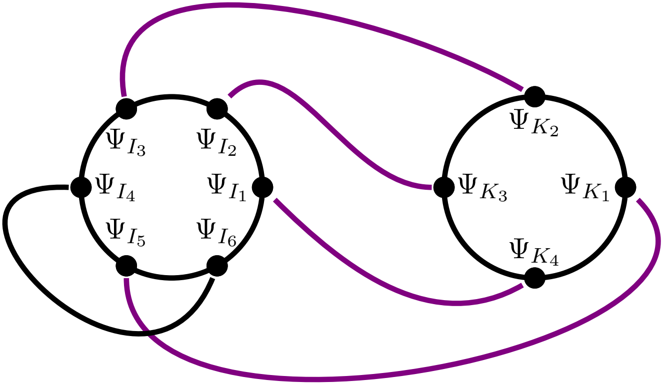

$\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_3}\Psi_{I_4}\Psi_{I_4}\Psi_{I_6}]$ and $\mathrm{tr}[\Psi_{I_1}\Psi_{I_2}\Psi_{I_3}\Psi_{I_6}]$ must be different from 0; hence in leading order all Ψ_I on each circle must be paired

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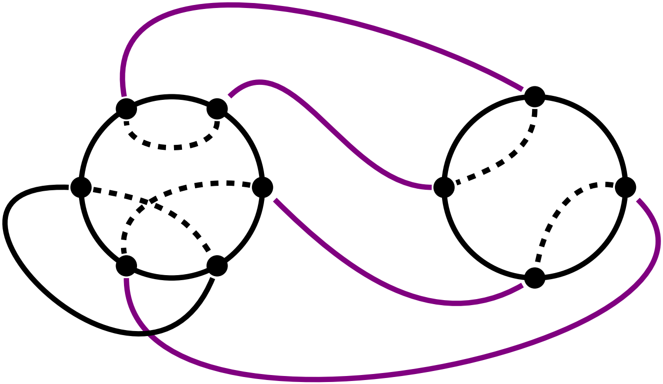

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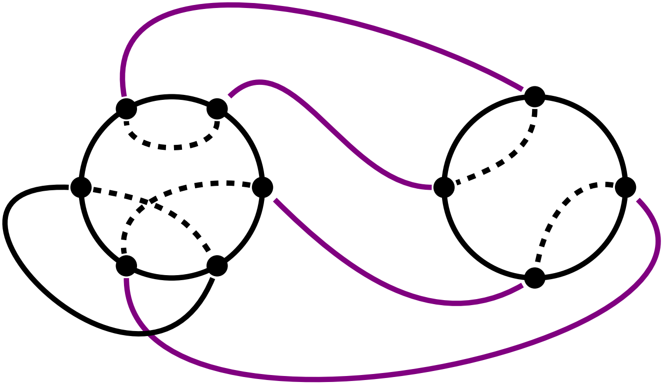

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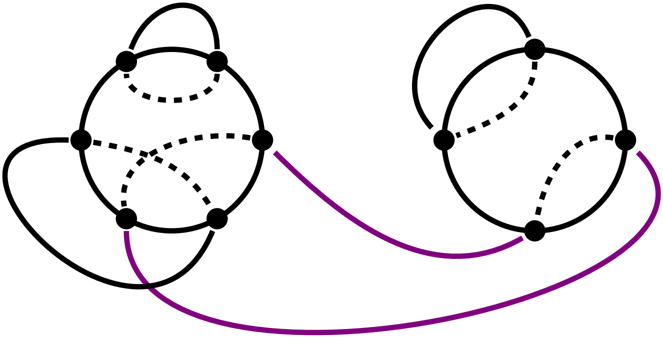
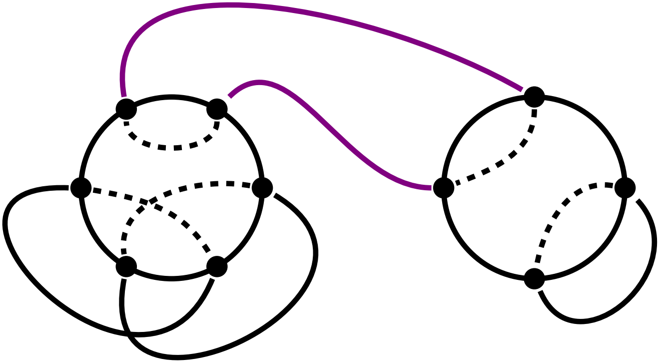
hence in leading order all Ψ_I on each circle must be paired
however, too many pairs are now identified, so this is subleading

Contribution to $k_2(\cdot,\cdot)\,\mathrm{tr}[\Psi_{I_1}\cdots\Psi_{I_6}]\,\mathrm{tr}[\Psi_{K_1}\cdots\Psi_{K_4}])$



leading order contributions are like this ...

Contribution to $k_2(\cdot,\cdot)\,\mathrm{tr}[\Psi_{I_1}\cdots\Psi_{I_6}]\,\mathrm{tr}[\Psi_{K_1}\cdots\Psi_{K_4}])$



leading order contributions are like this ... or like this ...

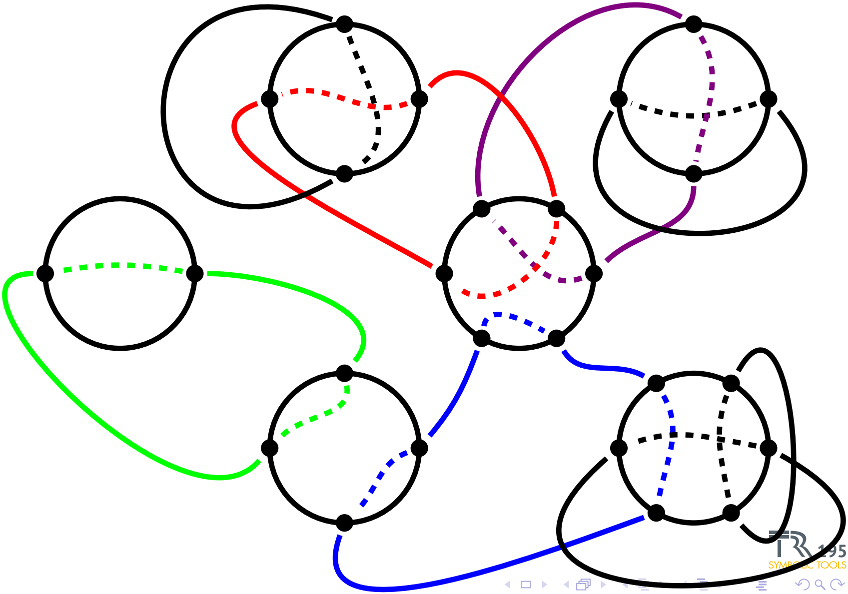
Let $(H_k)_{k\in\mathbb{N}}$ be independent copies of the SYK model H_{n,q_n} from with centered Gaussian random coefficients. For positive integers $m, r_1, \dots, r_m$ set $r := r_1 + \cdots + r_m$ and denote

$$\theta = \{T_1, \dots, T_m\},$$

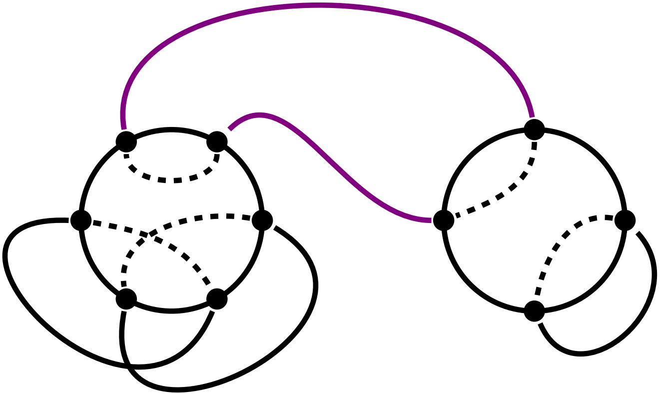
where $T_1 = [1, r_1], T_2 = [1 + r_1, r_1 + r_2], \dots, T_m = [1 + r_1 + \cdots + r_{m-1}, r]$. Given a function $\varepsilon : [1, k] \to \mathbb{N}$, let us denote for each $1 \leq i \leq m$ the functions $\varepsilon_i := \varepsilon|_{T_i}$.

$$\binom{n}{q_n}^{m-1} \cdot k_m\left(\mathrm{tr}(H_{\varepsilon_1}), \dots, \mathrm{tr}(H_{\varepsilon_m})\right) \xrightarrow{n\to\infty} \sum_{\substack{\pi,\pi'\in\mathcal{P}_2(r)\\ \pi\vee\theta=1_r,\pi'\leq\theta\\ \pi\leq\ker\varepsilon\\ |\pi\vee\pi'|=r/2-m+1}} q^{cr(\pi')},$$

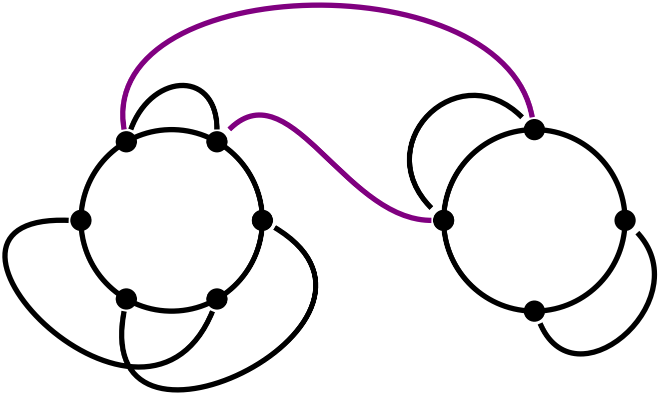
Contribution to 6-th Correlation Function



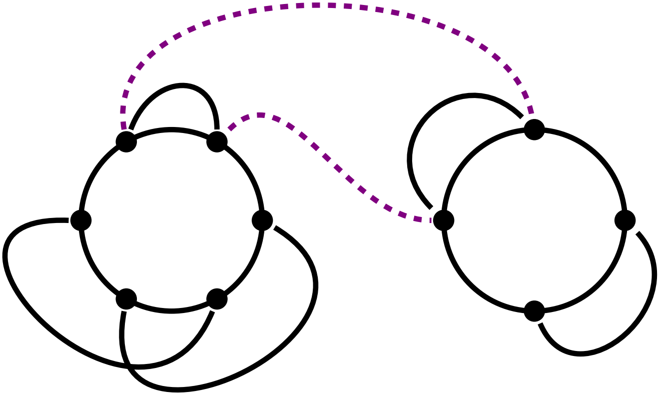
Leading Order Contribution in Fluctuations



Leading Order Contribution in Fluctuations



Leading Order Contribution in Fluctuations

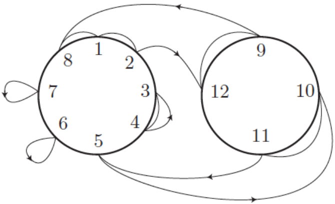


. . . looks like contribution in second order freeness . . .

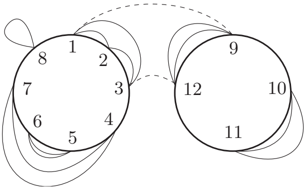
Second Order Freeness

In the sparse SYK random matrix model there are no analogues of first order contributions, only second order contributions show up . . .

first order contribution



second order contribution



Section 7

Summary

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- asymptotic expectation of SYK model converges to distribution of q-Brownian motion
- asymptotic higher order correlations show some (but not all) features of higher order freeness

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Thank You!