The emergence of time

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Part I

General introduction

Noncommutativity and modular time

Time as derived quantity

classical static space $\rightarrow$ no time

quantum space $\rightarrow$ quantum fluctuations

no static quantum space may exist

noncommutativity generates time

The arrow of time

The arrow of time is viewed both classically and in quantum physics

thermodynamics $\rightarrow$ positive entropy

quantum mechanics $\rightarrow$ collapse of the wave function

Known question: is there a general frame to encompass both?

Of course, we keep in mind that time is a relative concept as we learnt from Einstein.

Quantum Mechanics and Noncommutativity

Schrödinger → *von Neumann uniqueness* ← *Heisenberg*

- Schrödinger:

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t)$$

Differential equations

- Heisenberg:

$$PQ - QP = i\hbar I$$

Linear operators on Hilbert space, **noncommutativity is essential**!

Operator Algebras

Classical Commutative	Quantum Noncommutative
Manifold X $C^\infty(X)$	$*$-algebra A
Topological space X $C(X)$	C^*-algebra $\mathfrak{A}$
Measure space X $L^\infty(X, \mu)$	von Neumann algebra $\mathcal{A}$

Quantum calculus with infinitely many degrees of freedom

CLASSICAL	Classical variables Differential forms Chern classes	Variational calculus Infinite dimensional manifolds Functions spaces Wiener measure
QUANTUM	Quantum geometry Fredholm operators Index Cyclic cohomology	Subfactors Bimodules, Endomorphisms Multiplicative index Supersymmetric QFT, $(\mathfrak{A}, \mathcal{H}, Q)$

Thermal equilibrium states

A primary role in thermodynamics is played by the equilibrium distribution.

Gibbs states

Finite quantum system: $\mathfrak{A}$ matrix algebra with Hamiltonian H and evolution $\tau_t = \mathrm{Ad}e^{itH}$. Equilibrium state φ at inverse temperature β is given by the Gibbs property

$$\varphi(X) = \frac{\mathrm{Tr}(e^{-\beta H}X)}{\mathrm{Tr}(e^{-\beta H})}$$

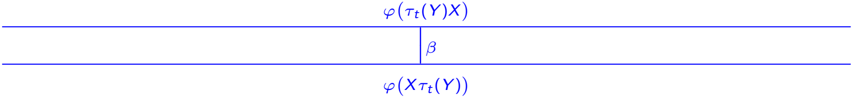
What are the equilibrium states at infinite volume where there is no trace, no inner Hamiltonian?

KMS states (HHW, Baton Rouge conference 1967)

Infinite volume. $\mathfrak{A}$ a C^*-algebra, τ a one-par. automorphism group of $\mathfrak{A}$. A state φ of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if for $X, Y \in \mathfrak{A}$ $\exists$ function F_{XY} s.t.

(a) $F_{XY}(t) = \varphi(X\tau_t(Y))$
(b) $F_{XY}(t + i\beta) = \varphi(\tau_t(Y)X)$

F_{XY} bounded analytic on $S_\beta = \{0 < \Im z < \beta\}$



KMS states generalise Gibbs states, equilibrium condition for infinite systems

Tomita-Takesaki modular theory

$\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$, $\varphi = (\Omega, \cdot\Omega)$ normal faithful state on $\mathcal{M}$. Embed $\mathcal{M}$ into $\mathcal{H}$

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow[\text{isometric}]{X \mapsto X^*} & \mathcal{M} \\
{\scriptstyle X \to X\Omega}\downarrow & & \downarrow{\scriptstyle X \to X\Omega} \\
\mathcal{H} & \xrightarrow[\text{non isometric}]{S_0 : X\Omega \mapsto X^*\Omega} & \mathcal{H}
\end{array}$$

$S = \bar{S}_0$, $\Delta = S^* S > 0$ positive selfadjoint

$$t \in \mathbb{R} \mapsto \sigma_t^{\varphi} \in \mathrm{Aut}(\mathcal{M})$$

$$\sigma_t^{\varphi}(X) = \Delta^{it} X \Delta^{-it}$$

intrinisic dynamics associated with φ (*modular automorphisms*).

Modular theory and temperature

By a remarkable historical accident, Tomita announced the theorem at the 1967 Baton Rouge conference. Soon later Takesaki completed the theory and charcterised the modular group by the KMS condition.

• σ^{φ} is a purely noncommutative object (trivial in the commutative case)

• it is a thermal equilibrium evolution If $\varphi(X) = \mathrm{Tr}(\rho X)$ (type I case) then $\sigma_t^{\varphi}(X) = \rho^{it} X \rho^{-it}$

• arrow of modular time is thermodynamical KMS condition at inverse temperature $\beta = -1$

• modular time is intrinsic modulo scaling the rescaled group $t \mapsto \sigma^{\varphi}_{-t/\beta}$ is physical, β^{-1} KMS temperature

Time as thermodynamical effect

quantum physics

$\updownarrow$

KMS with positive temperature

$\updownarrow$

modular time arrow

$\updownarrow$

positive entropy

$\updownarrow$

thermodynamical arrow

If time is the modular time, then the time arrow is associated both with positive entropy and with quantum structure!

Jones index

Factors (von Neumann algebras with trivial center) are “very infinite-dimensional” objects. For an inclusion of factors $\mathcal{N} \subset \mathcal{M}$ the Jones index $[\mathcal{M} : \mathcal{N}]$ measure the relative size of $\mathcal{N}$ in $\mathcal{M}$. Surprisingly, the index values are quantised:

$$[\mathcal{M} : \mathcal{N}] = 4\cos^2\left(\frac{\pi}{n}\right) \ , \quad n = 3, 4, \ldots \quad \text{or} \quad [\mathcal{M} : \mathcal{N}] \geq 4$$

Jones index appears in many places in math and in physics.

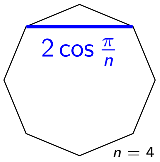


Figure: Jones index values

Quantum Field Theory

In QFT we have a quantum system with infinitely many degrees of freedom. The system is relativistic and there is particle creation and annihilation.

No mathematically rigorous QFT model with interaction still exists in 3+1 dimensions!

Haag local QFT:

$$O \text{ spacetime regions} \mapsto \text{von Neumann algebras } \mathcal{A}(O)$$

to each region one associates the "noncommutative functions" with support in O.

Local QFT nets

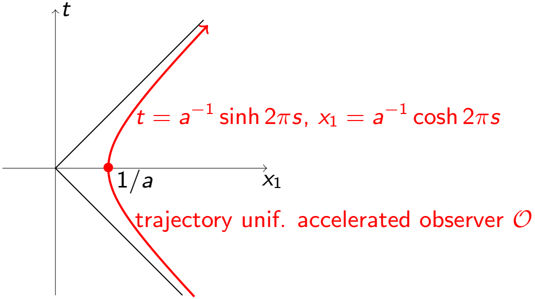
Local net $\mathcal{A}$ on spacetime M: map $O \subset M \mapsto \mathcal{A}(O) \subset B(\mathcal{H})$ s.t.

- *Isotony*, $O_1 \subset O_2 \implies \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- *Locality*, O_1, O_2 spacelike $\implies [\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$
- *Poincaré covariance* (conformal, diffeomorphism) .
- *Positive energy and vacuum vector*.

$O \mapsto \mathcal{A}(O)$: "Noncommutative chart" in QFT

Bisognano-Wichmann theorem '75, Sewell comment '80

Rindler spacetime (wedge $x_1 > 0$), vacuum modular group



a uniform acceleration of $\mathcal{O}$
s/a proper time of $\mathcal{O}$
$\beta = 2\pi/a$ inverse KMS temperature of $\mathcal{O}$

Hawking-Unruh effect!
Time is geodesic, quantum gravitational effect!

Part II

Applications

Intrinsic bounds on entropy

Entropy of finite systems

$X = \{x_1, \dots x_n\}$ a set of events. If x_i occurs with probability p_i, its information is $-\log p_i$

$$\text{Shannon entropy} : S(P) = -\sum p_i \log p_i \ .$$

If $Q = \{q_1, \dots q_n\}$ other probability distribution (state)

$$\text{Relative entropy} : S(P\|Q) = \sum p_i(\log p_i - \log q_i)$$

mean value in the state P of the difference between the information carried by the state P and the state Q.

Noncommutative entropy: $\varphi = -\mathrm{Tr}(\rho_\varphi \,\cdot)$ state on a matrix algebra

$$\text{von Neumann entropy} : S(\varphi) = \ -\mathrm{Tr}(\rho_\varphi \log \rho_\varphi)$$

Umegaki's relative entropy

$$S(\varphi\|\psi) =: \mathrm{Tr}\big(\rho_\varphi(\log \rho_\varphi - \log \rho_\psi)\big)$$

Araki's relative entropy

An infinite quantum system is described by a von Neumann algebra $\mathcal{M}$ typically not of type I so Tr does not exists; however Araki's relative entropy between two faithful normal states φ and ψ on $\mathcal{M}$ is defined in general by

$$S(\varphi|\psi) \equiv -(\eta, \log \Delta_{\xi,\eta}\, \eta)$$

where ξ, η are cyclic vector representatives of φ, ψ and $\Delta_{\xi,\eta}$ is the relative modular operator associated with ξ, η.

$$S(\varphi|\psi) \geq 0$$

positivity of the relative entropy

Relative entropy is one of the key concepts. We take the view that relative entropy is a primary concept and all entropy notions are derived concepts

CP maps, quantum channels and entropy

$\mathcal{N}, \mathcal{M}$ vN algebras. A linear map $\alpha : \mathcal{N} \to \mathcal{M}$ is completely positive if

$$\alpha \otimes \mathrm{id}_n : \mathcal{N} \otimes \mathrm{Mat}_n(\mathbb{C}) \to \mathcal{M} \otimes \mathrm{Mat}_n(\mathbb{C})$$

is positive $\forall n$ (quantum operation)

ω faithful normal state of $\mathcal{M}$ and $\alpha : \mathcal{N} \to \mathcal{M}$ CP map as above. Set

$$\mathrm{H}_\omega(\alpha) \equiv \sup_{(\omega_i)} \sum_i S(\omega|\omega_i) - S(\omega \cdot \alpha|\omega_i \cdot \alpha)$$

supremum over all ω_i with $\sum_i \omega_i = \omega$.

The (conditional) entropy $H(\alpha)$ of α is defined by

$$\mathrm{H}(\alpha) = \inf_\omega \mathrm{H}_\omega(\alpha)$$

infimum over all "full" states ω for α. Clearly $\mathrm{H}(\alpha) \geq 0$ because $\mathrm{H}_\omega(\alpha) \geq 0$ by the monotonicity of the relative entropy .

α is a quantum channel if its conditional entropy $\mathrm{H}(\alpha)$ is finite.

Generalisation of Stinespring dilation

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a normal, completely positive unital map between the vN algebras $\mathcal{N}$, $\mathcal{M}$. A pair (ρ, v) $\rho : \mathcal{N} \to \mathcal{M}$ a homomorphism, $v \in \mathcal{M}$ an isometry s.t.

$$\alpha(n) = v^* \rho(n) v \ , \quad n \in \mathcal{N} \ .$$

(ρ, v) is *minimal* if the left support of $\rho(\mathcal{N}) v \mathcal{H}$ is qual to 1.

Thm Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a normal, CP unital map with $\mathcal{N}$, $\mathcal{M}$ properly infinite. There exists a minimal dilation pair (ρ, v) for α. If (ρ_1, v_1) is another minimal pair, $\exists !$ unitary $u \in \mathcal{M}$ such that

$$u\rho(n) = \rho_1(n) u \, , \quad v_1 = uv, \qquad n \in \mathcal{N}$$

We have

$$H(\alpha) = \log Ind(\alpha) \qquad \text{(minimal index)}$$

Bimodules and CP maps

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a completely positive, normal, unital map and ω a faithful normal state of $\mathcal{M}$

$\exists!$ $\mathcal{N} - \mathcal{M}$ bimodule $\mathcal{H}_\alpha$, with a cyclic vector $\xi_\alpha \in \mathcal{H}$ and left and right actions ℓ_α and r_α, such that

$$(\xi_\alpha, \ell_\alpha(n)\xi_\alpha) = \omega_{\rm out}(n)\,, \quad (\xi_\alpha, r_\alpha(m)\xi_\alpha) = \omega_{\rm in}(m)\,,$$

with $\omega_{\rm in} \equiv \omega$, $\omega_{\rm out} \equiv \omega_{\rm in} \cdot \alpha$. Converse is true.

$$\text{CP map } \alpha \longleftrightarrow \text{cyclic bimodule } \mathcal{H}_\alpha$$

We have

$$H(\alpha) = \log Ind(\mathcal{H}_\alpha) \qquad \text{(Jones' index)}$$

Promoting modular theory to the bimodule setting

$\mathcal{H}$ an $\mathcal{N} - \mathcal{M}$-bimodule with finite Jones' index $\mathrm{Ind}(\mathcal{H})$

Given faithful, normal, states φ, ψ on $\mathcal{N}$ and $\mathcal{M}$, I define the modular operator $\Delta_{\mathcal{H}}(\varphi|\psi)$ of $\mathcal{H}$ with respect to φ, ψ as

$$\Delta_{\mathcal{H}}(\varphi|\psi) \equiv d(\varphi \cdot \ell^{-1})/d(\psi \cdot r^{-1} \cdot \varepsilon) \ ,$$

Connes' spatial derivative, $\varepsilon : \ell(\mathcal{N})' \to r(\mathcal{M})$ is the minimal conditional expectation

$\log \Delta_{\mathcal{H}}(\varphi|\psi)$ is called the modular Hamiltonian of the bimodule $\mathcal{H}$, or of the quantum channel α if $\mathcal{H}$ is associated with α.

Properties of the modular Hamiltonian

If $\mathcal{N}$, $\mathcal{M}$ factors

$$\Delta^{it}_{\mathcal{H}}(\varphi|\psi)\ell(n)\Delta^{-it}_{\mathcal{H}}(\varphi|\psi) = \ell\big(\sigma^{\varphi}_{t}(n)\big)$$

$$\Delta^{it}_{\mathcal{H}}(\varphi|\psi)r(m)\Delta^{-it}_{\mathcal{H}}(\varphi|\psi) = r\big(\sigma^{\psi}_{t}(m)\big)$$

(implements the dynamics)

$$\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2) \otimes \Delta^{it}_{\mathcal{K}}(\varphi_2|\varphi_3) = \Delta^{it}_{\mathcal{H}\otimes\mathcal{K}}(\varphi_1|\varphi_3)$$

(additivity of the energy)

$$\Delta^{it}_{\bar{\mathcal{H}}}(\varphi_2|\varphi_1) = d_{\mathcal{H}}^{-i2t}\,\overline{\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2)}$$

If $T : \mathcal{H} \to \mathcal{H}'$ is a bimodule intertwiner, then

$$T\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2) = (d_{\mathcal{H}'}/d_{\mathcal{H}})^{it}\Delta^{it}_{\mathcal{H}'}(\varphi_1|\varphi_2)T$$

Connes's bimodule tensor product w.r.t. φ_2; $d_{\mathcal{H}} = \sqrt{Ind(\mathcal{H})}$

Physical Hamiltonian

We may modify the modular Hamiltonian in order to fulfil the right physical requirements (additivity of energy, invariance under charge conjugation,...)

$$K(\varphi_1|\varphi_2) = -\log \Delta_{\mathcal{H}}(\varphi_1|\varphi_2) - \log d$$

is the physical Hamiltonian (at inverse temperature 1).

The physical Hamiltonian at inverse temperature $\beta > 0$ is given by

$$-\beta^{-1} \log \Delta - \beta^{-1} \log d$$

From the modular Hamiltonian to the physical Hamiltonian:

$$-\log \Delta \xrightarrow{\text{shifting}} -\log \Delta - \log d \xrightarrow{\text{scaling}} \beta^{-1}\big(-\log \Delta - \log d\big)$$

The shifting is intrinsic, the scaling is to be determined by the context!

Properties of the physical Hamiltonian

- $U_{\mathcal{H}}(t) = e^{itK_{\mathcal{H}}}$ *implements the dynamics:*

$$U_{\mathcal{H}}(t)(\varphi|\psi)\ell(n)U_{\mathcal{H}}(-t)(\varphi|\psi) = \ell\big(\sigma_t^{\varphi}(n)\big)$$

$$U_{\mathcal{H}}(t)(\varphi|\psi)r(m)U_{\mathcal{H}}(-t)(\varphi|\psi) = r\big(\sigma_t^{\psi}(m)\big)$$

- *additivity for composition:* (Connes's bimodule tensor product)

$$U_{\mathcal{H}}(t)(\varphi_1|\varphi_2) \otimes U_{\mathcal{K}}(t)(\varphi_2|\varphi_3) = U_{\mathcal{H}\otimes\mathcal{K}}(t)(\varphi_1|\varphi_3)$$

- *charge symmetry:*

$$U_{\bar{\mathcal{H}}}(t)(\varphi_2|\varphi_1) = \overline{U_{\mathcal{H}}(t)}$$

- *additivity for disjoint systems:*

If $T : \mathcal{H} \to \mathcal{H}'$ is a bimodule intertwiner, then

$$TU_{\mathcal{H}}(t)(\varphi_1|\varphi_2) = U_{\mathcal{H}'}(t)(\varphi_1|\varphi_2)T$$

Modular and Physical Hamiltonians for a quantum channel

We now are going to compare two states of a physical system, $\omega_{\rm in}$ is a suitable reference state, e.g. the vacuum in QFT, and $\omega_{\rm out}$ is a state that can be reached from $\omega_{\rm in}$ by some physically realisable process (quantum channel).

$\alpha : \mathcal{N} \to \mathcal{M}$ be a quantum channel (normal, unital CP map with finite entropy) and $\omega_{\rm in}$ a faithful normal state of $\mathcal{M}$. $\omega_{\rm out} = \omega_{\rm in} \cdot \alpha$

$$\log \Delta_\alpha \equiv \log \Delta_{\mathcal{H}_\alpha}$$

$$K_\alpha = \beta^{-1} K_{\mathcal{H}_\alpha} = \beta^{-1}\big(- \log \Delta_{\mathcal{H}_\alpha} - \log d_{\mathcal{H}_\alpha}\big)$$

(physical Hamiltonian at inverse temperature β)

K_α may be considered as a local Hamiltonian associated with α and the state transfer with input state $\omega_{\rm in}$.

Thermodynamical quantities

The entropy $S \equiv S_{\alpha,\omega_{\rm in}}$ of α is

$$S = -(\hat{\xi}, \log \Delta_\alpha \hat{\xi})$$

where $\hat{\xi}$ is a vector representative of the state $\omega_{\rm in} \cdot r^{-1} \cdot \varepsilon$ in $\mathcal{H}_\alpha$.

The quantity

$$E = (\hat{\xi}, K\hat{\xi})$$

is the relative energy w.r.t. the states $\omega_{\rm in}$ and $\omega_{\rm out}$.

The free energy F is now defined by the relative partition function

$$F = -\beta^{-1} \log(\hat{\xi}, e^{-\beta K}\hat{\xi})$$

F satisfies the thermodynamical relation

$$F = E - TS$$

Bekenstein's bound

For decades, modular theory has played a central role in the operator algebraic approach to QFT, very recently several physical papers in other QFT settings are dealing with the modular group, although often in a heuristic (yet powerful) way!

I will discuss the Bekenstein bound, a universal limit on the entropy that can be contained in a physical system with given size and given total energy

If R is the radius of a sphere that can enclose our system, while E is its total energy including any rest masses, then its entropy S is bounded by

$$S \leq \lambda RE$$

The constant λ is often proposed $\lambda = 2\pi$ (natural units).

A form of Bekenstein bound

As $F = \frac{1}{2}\beta^{-1}\mathrm{H}(\alpha)$, we have

$$F \geq 0 \quad \text{(positivity of the free energy)}$$

because

$$\mathrm{H}(\alpha) \geq 0 \quad \text{(monotonicity of the entropy)}$$

So the above thermodynamical relation

$$F = E - \beta^{-1}S$$

entails the following general, rigorous version of the Bekenstein bound

$$S \leq \beta E$$

To determine β we have to plug this general formula in a physical context

Landauer's bound for infinite systems

Landauer's principle: *any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information bearing degrees of freedom of the information processing apparatus or its environment* (cf. C. Bennet)

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a quantum channel between quantum systems $\mathcal{N}$, $\mathcal{M}$. based on the thermodynamical relation $dF = dE - TdS$, the free energy is

$$F_\alpha \equiv \left\langle K_{\mathcal{H}}(\varphi_1|\varphi_2) + \beta^{-1} log \Delta_{\mathcal{H}}(\varphi_1|\varphi_2) \right\rangle = \beta^{-1} \log d \geq 0$$

If α is irreversible, then

$$F_\alpha \geq \frac{1}{2} kT \log 2$$

The *original lower bound* for the incremental free energy is

$$F_\alpha \geq kT \log 2$$

it remains true for finite-dim. systems $\mathcal{N}$, $\mathcal{M}$.

Part III

QFT information and modular theory

Entropy density

The information carried by a classical wave

Suppose that Alice encodes and sends information by an undulatory signal, what information can Bob get by the wave packet in a given region at later time?

By a wave (or wave packet), we mean a real solution of the Klein-Gordon equation

$$(\Box + m^2)\Phi = 0 \ ,$$

with compactly supported, smooth Cauchy data $\Phi|_{x^0=0}$, $\Phi'|_{x^0=0}$.

Classical field theory describes Φ by the stress-energy tensor $T_{\mu\nu}$, that provides the energy-momentum density of Φ at any time.

But, how to define the information, or entropy, carried by Φ in a given region at a given time?

We give a classical answer to such a classical question by Operator Algebras and Quantum Field Theory

Standard subspaces

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a closed, real linear subspace.
Symplectic complement:

$$H' = \{\xi \in H : \Im(\xi, \eta) = 0 \ \forall \eta \in H\}$$

H is *cyclic* if $\overline{H + iH} = \mathcal{H}$ and *separating* if $H \cap iH = \{0\}$.

A standard subspace H of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ which is both cyclic and separating. H is standard iff H' is standard.

H standard subspace $\rightarrow$ anti-linear operator $S : D(S) \subset \mathcal{H} \rightarrow \mathcal{H}$,

$$S : \xi + i\eta \rightarrow \xi - i\eta, \ \xi, \eta \in H$$

$S^2 = 1|_{D(S)}$. S is closed and densely defined, indeed

$$S_H^* = S_{H'}$$

Modular theory for standard subspaces

Set $S = J\Delta^{1/2}$, polar decomposition of $S = S_H$.

Then J is an anti-unitary involution, $\Delta > 0$ is non-singular and $J\Delta J = \Delta^{-1}$.

$$\Delta^{it} H = H, \quad JH = H'$$

(one particle Tomita-Takesaki theorem).

Leyland, Roberts, Testard (unpublished) in this form

Entropy of a vector relative to a real linear subspace

Our analysis relies on the concept of entropy S_k of a vector k in a Hilbert space $\mathcal{H}$ with respect to a real linear subspace H of $\mathcal{H}$. Our formula for the entropy of k with respect to H is

$$S_k = \Im(k, P_H i \log \Delta\, k)$$

Here P_H is the crucial cutting projection $P_H : H + H' \to H$

$$P_H : h + h' \mapsto h$$

in terms of J and Δ,

$$P_H = \Delta^{-1/2}(\Delta^{-1/2} - \Delta^{1/2})^{-1} + J(\Delta^{-1/2} - \Delta^{1/2})^{-1}$$

Entropy of a wave

Waves' time-independent symplectic form

$$\frac{1}{2}\int_{x^0=t}(\Phi'\Psi - \Psi'\Phi)dx \ ,$$

The symplectic form is the imaginary part of complex Hilbert space scalar product (that depends on the mass).

Waves with Cauchy data supported in the half-space $x^1 \geq 0$ form a real linear subspace $H(W)$ (W wedge).

The entropy S_Φ of Φ is the entropy of the vector Φ w.r.t. $H(W)$

By the Bisognano-Wichmann theorem, we have the modular group of $H(W)$

Entropy of a wave

Let Φ be a real Klein-Gordon wave and $H = H(W_\lambda)$. W_λ null translated wedge.

The entropy $S_\Phi(\lambda)$ of Φ w.r.t. the wedge region W_λ is the entropy of the vector Φ w.r.t. the standard subspace $H(W_\lambda)$.

$$S_\Phi(\lambda) = 2\pi \int_{x^0=\lambda,\, x^1 \geq \lambda} (x^1 - \lambda)\, T_{00}(x)\, dx$$

then

$$S''_\Phi(\lambda) = 2\pi \int_{x^0=\lambda,\, x^1=\lambda} \langle v, Tv \rangle dx \geq 0 \ ,$$

where v is the light-like vector $v = (1, 1, 0 \ldots, 0)$ *QNEC for coherent states, constant deformations*. (Work with F. Ciolli, G. Ruzzi.)

Here the energy density is

$$\langle T_{00} \rangle_\Phi = \frac{1}{2}\big(\Phi'^2 + |\nabla\Phi|^2 + m^2\Phi^2\big)$$

A local massive Hamiltonian

Work in progress/Conjecture. The modular Hamiltonian $\log \Delta_B$ associated with the unit ball B in the free scalar, mass m QFT is (on Cauchy data)

$$-2\pi A_m = \log \Delta_B \,.$$

$$\log \Delta_B = 2\pi \imath_m \begin{bmatrix} 0 & \frac{1}{2}(1-r^2) \\ \frac{1}{2}(1-r^2)(\nabla^2 - m^2) - r\partial_r - D - \frac{1}{2}m^2 G_m^B & 0 \end{bmatrix}$$

with L_m the massive Legendre operator

$$L_m = \frac{1}{2}(1-r^2)(\nabla^2 - m^2) - r\partial_r - D$$

and G_m^B is the Green integral operator with Yukawa potential

$$G_m^B f(\mathbf{x}) = \frac{1}{4\pi}\int_B \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} f(\mathbf{y}) d\mathbf{y} \,,$$

Work in progress with G. Morsella. *The conjecture is true if $m = 0$.*

Local information in a wave packet

With $S_\Phi(R)$ the entropy of Φ in the radius R ball cantered at $\bar{\mathbf{x}}$, we have

$$S_\Phi(R) = \pi \int_{B_R(\bar{\mathbf{x}})} \frac{R^2 - r^2}{R} \langle T_{00}^{(m)}(t,\mathbf{x}) \rangle_\Phi \, d\mathbf{x} \qquad \text{stress-energy tensor term}$$

$$+ \pi \frac{d-1}{2R} \int_{B_R(\bar{\mathbf{x}})} \Phi^2(t,\mathbf{x}) \, d\mathbf{x} \qquad \text{mass independent normalisation}$$

$$+ \pi \frac{m^2}{R} \int_{B_R(\bar{\mathbf{x}})} G_m(\mathbf{x}-\mathbf{y}) \Phi^2(t,\mathbf{x}) \, d\mathbf{x} \, d\mathbf{y} \qquad \text{Yukawa potential term}$$

with $r = |\mathbf{x} - \bar{\mathbf{x}}|$ a

Yukawa potential

Yukawa predicted the potential for an exchange particle for the neutron-proton interaction:

$$G_m = -g^2 \frac{e^{-\alpha m r}}{r}$$

g scaling constant, m mass, r radial distance to the particle, and α constant s.t. $r = 1/\alpha m$ is the approximate range.

This particle was later found, the *pion*, and Yukawa was awarded the Nobel prize.

The parabolic distribution

$$S_\Phi(R) = \pi \int_{B_R(\bar{\mathbf{x}})} \underbrace{\frac{R^2 - r^2}{R}}_{parab.\ distr.} \langle T_{00}^{(m)}(t, \mathbf{x}) \rangle_\Phi d\mathbf{x} + \cdots$$

The parabolic distribution is higher-dimensional generalization of Wigner semi-circular distribution in three dimensional space (the marginal distribution function of a spherical distribution)

As $R \to 0$

$$S_\Phi(R, \mathbf{x}) = \frac{\pi}{d} \left(\langle T_{00} \rangle_\Phi(t, \mathbf{x}) + D\Phi^2(t, \mathbf{x}) \right) A_{d-1}(R) + ...$$

Here $A_{d-1}(R) = 2\frac{\pi^{d/2}}{\Gamma(d/2)} R^{d-1}$ is the area of boundary sphere ∂B_R, cf. holographic area theorems, black holes (Bekenstein) and other contexts.