Preservers of totally positive kernels and Polya frequency functions

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Contents

Part 1: Euclidean metric

Distance geometry

Let (X, ρ) be a metric space. Describe operations $\phi(\rho)$ which may be performed on the metric and which enhance various properties of the topological space X.

For instance: $\rho/(\rho + 1)$ and ρ^γ, if $\gamma \in (0, 1)$, also satisfy the axioms of a metric, with the former making it bounded.

Blumenthal, 1953. The new metric space (X, ρ^γ) has the four-point property if $\gamma \in (0, 1/2]$: every four-point subset of X can be embedded isometrically into Euclidean space.

Embedding into a real Hilbert space

Menger, Fréchet, Schoenberg, 1935. Let $d \geq 1$ be an integer and let (X, ρ) be a metric space. An $(n+1)$-tuple of points x_0, x_1, ..., x_n in X can be isometrically embedded into Euclidean space $\mathbb{R}^d$, but not into $\mathbb{R}^{d-1}$, if and only if the matrix

$$[\rho(x_0, x_j)^2 + \rho(x_0, x_k)^2 - \rho(x_j, x_k)^2]_{j,k=1}^n,$$

is positive semidefinite with rank equal to d.

Corollary

A separable metric space (X, ρ) can be isometrically embedded into Hilbert space if and only if, for every $(n+1)$-tuple of points $(x_0, x_1, \ldots, x_n)$ in X, where $n \geq 2$, the matrix

$$[\rho(x_0, x_j)^2 + \rho(x_0, x_k)^2 - \rho(x_j, x_k)^2]_{j,k=1}^{n}$$

is positive semi-definite.

Embedding into spheres

Schoenberg, 1935. Let (X, ρ) be a metric space and let $(x_1, \dots, x_n)$ be an n-tuple of points in X. For any integer $d \geq 2$, there exists an isometric embedding of $(x_1, \dots, x_n)$ into S^{d-1} but not S^{d-2} if and only if

$$\rho(x_j, x_k) \leq \pi \qquad (1 \leq j, k \leq n)$$

and the matrix $\left[\cos \rho(x_j, x_k)\right]_{j,k=1}^{n}$ is positive semidefinite of rank d.

Distance transforms

Schoenberg, 1938. A separable metric space (X, ρ) can be embedded isometrically into Hilbert space if and only if the kernel

$$X \times X \to (0, \infty); \ (x, y) \mapsto \exp(-\lambda^2 \rho(x, y)^2)$$

is positive semidefinite for all $\lambda \in \mathbb{R}$.

Fractional powers

From

$$\xi^\alpha = c_\alpha \int_0^\infty (1 - e^{-s^2\xi^2}) s^{-1-\alpha} ds \qquad (\xi > 0,\ 0 < \alpha < 2),$$

where c_α is a normalization constant. Consequently,

$$\|x - y\|^\alpha = c_\alpha \int_0^\infty (1 - e^{-s^2\|x-y\|^2}) s^{-1-\alpha} ds.$$

one finds:

Let H be a Hilbert space with norm $\|\cdot\|$. For every $\delta \in (0,1)$, the metric space $(H, \|\cdot\|^\delta)$ is isometric to a subspace of a Hilbert space.

A real continuous function ϕ is called *positive definite in Euclidean space* $\mathbb{R}^d$ if the translation invariant kernel

$$(x, y) \mapsto \phi\big(\|x - y\|\big)$$

is positive semidefinite. Bochner's theorem and the rotation-invariance of this kernel prove that such a function ϕ is characterized by the representation

$$\phi(t) = \int_0^\infty \Omega_d(tu) d\mu(u),$$

where μ is a positive measure and

$$\Omega_d\big(\|x\|\big) = \int_{\|\xi\|=1} e^{ix\cdot\xi} d\sigma(\xi),$$

with σ the normalized area measure on the unit sphere in $\mathbb{R}^d$.

Transforms of distance function

Schoenberg- von Neumann, 1941.

Let H be a separable Hilbert space with norm $\|\cdot\|$.

1. For any integers $n \geq d > 1$, the metric space $(\mathbb{R}^d, \phi(\|\cdot\|))$ may be isometrically embedded into $(\mathbb{R}^n, \|\cdot\|)$ if and only if $\phi(t) = ct$ for some $c > 0$.
2. The metric space $(\mathbb{R}^d, \phi(\|\cdot\|))$ may be isometrically embedded into H if and only if

$$\phi(t)^2 = \int_0^\infty \frac{1 - \Omega_d(tu)}{u^2} d\mu(u),$$

where μ is a positive measure on the semi-axis such that

$$\int_1^\infty \frac{1}{u^2} \, \mathrm{d}\mu(u) < \infty.$$

3. The metric space $(H, \phi(\|\cdot\|))$ may be isometrically embedded into H if and only if

$$\phi(t)^2 = \int_0^\infty \frac{1 - e^{-t^2 u}}{u} d\mu(u),$$

where μ is a positive measure on the semi-axis such that

$$\int_1^\infty \frac{1}{u} d\mu(u) < \infty.$$

Screw lines in Hilbert space

The metric space $(\mathbb{R}, \phi(|\cdot|))$ isometrically embeds into Hilbert space if and only if

$$\phi(t)^2 = \int_0^\infty \frac{\sin^2(tu)}{u^2} d\mu(u) \qquad (t \in \mathbb{R}),$$

where μ is a positive measure on $\mathbb{R}_+$ satisfying

$$\int_1^\infty \frac{1}{u^2} d\mu(u) < \infty.$$

Khinchin, Kolmogorov, 1934-1940. Relevant to the theory of stationary stochastic processes.

Part 2: Fourier-Laplace transforms

Homogeneous spaces

Bochner, 1941. Let X be a compact homogeneous space. A continuous *invariant* function ρ on $X \times X$ is a Hilbert distance if and only if there exists a continuous, real-valued, invariant, positive definite kernel h on X and a point $x_0 \in X$, such that

$$\rho(x, y) = \sqrt{h(x_0, x_0) - h(x, y)} \qquad (x, y \in X).$$

Example 1D

Let $X = \mathbb{T} = \{e^{\mathrm{i}\theta} : \theta \in \mathbb{R}\}$ be the unit torus, endowed with the invariant arc-length measure. A continuous positive definite function $h : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ admits a Fourier decomposition

$$h(e^{\mathrm{i}x}, e^{\mathrm{i}y}) = \sum_{j,k \in \mathbb{Z}} a_{jk} e^{\mathrm{i}jx} e^{-\mathrm{i}ky}.$$

If h is further required to be rotation invariant, we find that

$$h(e^{\mathrm{i}x}, e^{\mathrm{i}y}) = \sum_{k \in \mathbb{Z}} a_k e^{\mathrm{i}k(x-y)},$$

where $a_k \geq 0$ for all $k \in \mathbb{Z}$ and $a_k = a_{-k}$ because h takes real values.

The series is Abel summable: $\sum_{k=0}^{\infty} a_k = h(1,1) < \infty$. Therefore, a rotation-invariant Hilbert distance ρ on the torus has the expression (after taking its square):

$$\begin{aligned}\rho(e^{\mathrm{i}x}, e^{\mathrm{i}y})^2 = h(1,1) - h(e^{\mathrm{i}x}, e^{\mathrm{i}y}) &= \sum_{k=1}^{\infty} a_k (2 - e^{\mathrm{i}k(x-y)} - e^{-\mathrm{i}k(x-y)}) \\ &= 2\sum_{k=1}^{\infty} a_k (1 - \cos k(x-y)) \\ &= 4\sum_{k=1}^{\infty} a_k \sin^2(k(x-y)/2).\end{aligned}$$

These are the periodic screw lines singled out by von Neumann and Schoenberg.

The sphere

Schoenberg, 1942. A continuous, real-valued, rotationally invariant and positive definite kernel f on the sphere S^{d-1} has a distinguished Fourier-series decomposition with non-negative coefficients. Specifically,

$$f(\cos\theta) = \sum_{k=0}^{\infty} c_k P_k^{(\lambda)}(\cos\theta)$$

where $\lambda = (d-2)/2$, $P_k^{(\lambda)}$ are the ultraspherical orthogonal polynomials, $c_k \geq 0$ for all $k \geq 0$ and $\sum_{k=0}^{\infty} c_k < \infty$.

The simplification due to passing to infinity

A real-valued function $f(\cos\theta)$ is positive definite on all spheres, independent of their dimension, if and only if

$$f(\cos\theta) = \sum_{k=0}^{\infty} c_k \cos^k\theta, \tag{1}$$

where $c_k \geq 0$ for all $k \geq 0$ and $\sum_{k=0}^{\infty} c_k < \infty$.

Hadamard-Schur calculus of symmetric matrices

Schoenberg, 1942. Let $f : [-1,1] \to \mathbb{R}$ be a continuous function. If the matrix $[f(a_{jk})]_{j,k=1}^n$ is positive semidefinite for all $n \geq 1$ and all positive semidefinite matrices $[a_{jk}]_{j,k=1}^n$ with entries in $[-1,1]$, then, and only then,

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \qquad (x \in [-1,1]),$$

where $c_k \geq 0$ for all $k \geq 0$ and $\sum_{k=0}^{\infty} c_k < \infty$.

Operations on Fourier transforms

Consider Wiener's algebra $W := \widehat{L^1(\mathbb{Z})} \subset C(\mathbb{T})$ of Fourier transforms of L^1 sequences.

Gelfand, 1941. Let $\phi \in W$ and let $f(z)$ be an analytic function defined in a neighborhood of $\phi(\mathbb{T})$. Then $f(\phi) \in W$.

Analyticity of such self-maps

Helson–Kahane–Katznelson–Rudin, 1959.
Let G be a locally compact abelian group and let Γ denote its dual, and suppose both are endowed with their respective Haar measures. Let $f : [-1, 1] \to \mathbb{C}$ be a function satisfying $f(0) = 0$.

1. If Γ is discrete and f operates on $\widehat{L^1(G)}$, then f is analytic in some neighborhood of the origin.
2. If Γ is not discrete and f operates on $\widehat{L^1(G)}$, then f is analytic in $[-1, 1]$.
3. If Γ is not compact and f operates on $\widehat{M(G)}$, then f can be extended to an entire function.

Pólya frequency functions

Schoenberg, 1947. A function $\Lambda : \mathbb{R} \to [0, \infty)$ is a *Pólya frequency function* if it is Lebesgue integrable and non-zero at two or more points, and the Toeplitz kernel

$$T_\Lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ (x, y) \mapsto \Lambda(x - y)$$

is *totally non-negative*: for any integer $p \geq 1$ and real numbers

$$x_1 < \cdots < x_p \qquad \text{and} \qquad y_1 < \cdots < y_p,$$

the matrix $\big(\Lambda(x_j - y_k)\big)_{j,k=1}^{p}$ has non-negative determinant.

Classical examples

For any $\kappa > 0$,

$$G_\kappa : \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ (x, y) \mapsto \exp\bigl(-\kappa(x-y)^2\bigr)$$

and

$$L_\kappa : \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ (x, y) \mapsto \exp\bigl(-\kappa|x-y|\bigr)$$

are totally non-negative kernels.

Laplace-transform characterization of PFFs

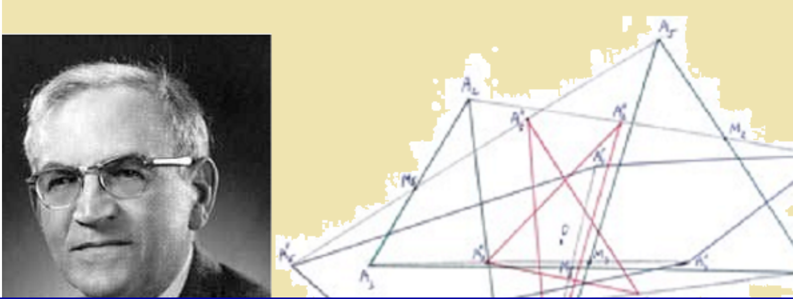
The bilateral Laplace transform

$$\mathcal{B}\{\Lambda\}(s) := \int_{\mathbb{R}} e^{-xs}\Lambda(x)\,\mathrm{d}x$$

of a Pólya frequency function Λ converges in a open strip containing the imaginary axis, and equals on this strip the reciprocal of an entire function Ψ in the Laguerre–Pólya class (consisting of *entire functions which are locally the limit of a series of polynomials whose roots are all real*).

I. J. Schoenberg dictionary

Conversely, any function Ψ of this form agrees with the reciprocal of the bilateral Laplace transform of some Pólya frequency function on its strip of convergence. The origin of the theory of splines.



One-sided Pólya frequency functions

If the one-sided Pólya frequency function Λ vanishes on $(-\infty, 0)$ then the reciprocal $1/\mathcal{B}\{\Lambda\}$ is the restriction of an entire function Ψ in the (first) Laguerre–Pólya class, so has the form

$$\Psi(s) = Ce^{\delta s} \prod_{j=1}^{\infty} (1 + \alpha_j s)$$

where

$$(C > 0,\ \delta \geq 0,\ \alpha_j \geq 0 \text{ and } 0 < \textstyle\sum_{j=1}^{\infty} \alpha_j < \infty).$$

Conversely, if the entire function Ψ has this form then there exists a Pólya frequency function Λ vanishing on $(-\infty, 0)$ such that $\Psi(s) = 1/\mathcal{B}\{\Lambda\}(s)$ on an open strip containing the origin.

Such a Pólya frequency function Λ is continuous and positive on (δ, ∞) and vanishes on $(-\infty, \delta)$. The function Λ is smooth if and only if α_j is non-zero for infinitely many j, and is continuous unless α_j is non-zero for exactly one j.

Part 3: Hirschman-Widder densities

Finitely determined Pólya frequency functions

We focus on the one-sided Pólya frequency functions which are continuous but non-smooth, that is *finitely determined* Pólya frequency functions of the form $\Lambda_{\boldsymbol{\alpha}}$, such that

$$\mathcal{B}\{\Lambda_{\boldsymbol{\alpha}}\}(s) = \prod_{j=1}^{m}(1+\alpha_j s)^{-1}, \quad \text{where } \boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_m) \text{ and } m \geq 2.$$

Historical context

In 1947, Schoenberg announced the notion of a Pólya frequency function and explored their basic properties.

In their 1949 work, Hirschman and Widder studied functions of the form Λ_{α} for distinct positive $\alpha_1, \ldots, \alpha_m$ and their degree of smoothness, via the Laplace transform.

This was followed by Schoenberg's first full paper on Pólya frequency functions in 1951. In this work, Schoenberg placed the analysis of Hirschman and Widder in a wider context.

Hirschman and Widder's 1955 monograph contains a detailed analysis of these functions and their Laplace transforms, and shows the relevance of such functions to operational calculus and approximation theory.

Hirschman–Widder densities

Let $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$ denote the set of non-zero real numbers. Given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{R}^\times)^m$, where $m \geq 2$, the corresponding *Hirschman–Widder density* is the unique continuous function

$$\Lambda_{\boldsymbol{\alpha}} : \mathbb{R} \to [0, \infty)$$

with bilateral Laplace transform

$$\int_{\mathbb{R}} e^{-xs} \Lambda_{\boldsymbol{\alpha}}(x) \, \mathrm{d}x = \prod_{j=1}^{m} (1 + \alpha_j s)^{-1}$$

on the open half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > -\alpha_j^{-1} \text{ for } j = 1, \ldots, m\}$.

Basic properties of HW densities

1. Each such function $\Lambda_{\boldsymbol{\alpha}}$ exists and is unique.
2. The function $\Lambda_{\boldsymbol{\alpha}}$ is both a Pólya frequency function and a probability density function. It is one sided if and only if all the entries of $\boldsymbol{\alpha}$ have the same sign.
3. The function $\Lambda_{\boldsymbol{\alpha}}$ has a multiplicative representation via convolution. The class of Pólya frequency functions and its subclass of Hirschman–Widder densities are both semigroups for the convolution product.
4. Given any $\boldsymbol{\alpha}$ as above, there is a unique m-tuple of real polynomials $\mathbf{c} = (c_1, \ldots, c_m)$ such that

$$\Lambda_{\boldsymbol{\alpha}}(x) \equiv \Lambda_{\mathbf{a},\mathbf{c}}(x) := \mathbf{1}_{x \geq 0} \sum_{j=1}^{m} c_j(x) e^{-a_j x},$$

Probabilistic interpretation

Let $\alpha \in (0, \infty)^m$, where $m \geq 2$. The Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$ is the probability density function for the random variable

$$\alpha_1 X_1 + \cdots + \alpha_m X_m,$$

where X_1, ..., X_m are independent and identically distributed exponential random variables with mean 1.

Indeed, X is an exponential random variable with mean 1 and $\alpha > 0$ then αX has density function $\varphi_\alpha := x \mapsto \mathbf{1}_{x \geq 0}\, \alpha^{-1} e^{-\alpha^{-1} x}$, and

$$\Lambda_{\boldsymbol{\alpha}} = \varphi_{\alpha_1} * \cdots * \varphi_{\alpha_m}.$$

First probabilistic interpretation

Hirschman–Widder density functions are studied in the probability and statistics literature under the name of *hypoexponential densities*. They are intimately connected to the time to absorption for finite-state Markov chains.

When the entries of α are equal, then Λ_{α} is an *Erlang density*, named after the father of queueing theory; this is a special case of the gamma distribution that occurs when the shape parameter is an integer.

These densities have found use in diverse applied fields, including queueing theory, population genetics, reliability analysis and cell biology.

A determinantal representation

Given $\mathbf{a} \in \mathbb{R}^m$, where $m \geq 2$, let the Vandermonde determinant

$$V(\mathbf{a}) = V(a_1, \ldots, a_m) := \det(a_j^{k-1}) = \prod_{1 \leq j < k \leq m} (a_k - a_j).$$

Let $\boldsymbol{\alpha}$ be distinct and $\mathbf{a}$ be the tuple of their reciprocals:

$$\Lambda_{\boldsymbol{\alpha}}(x) = \frac{a_1 \cdots a_m}{V(\mathbf{a})} \det \begin{pmatrix} e^{-a_1 x} & e^{-a_2 x} & e^{-a_3 x} & \cdots & e^{-a_m x} \\ 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{m-2} & a_2^{m-2} & a_3^{m-2} & \cdots & a_m^{m-2} \end{pmatrix} \qquad (x \geq 0).$$

Schur polynomials

Given a field $\mathbb{F}$ of size at least $m \geq 2$, and a tuple $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ of non-negative integers $\lambda_1 \leq \cdots \leq \lambda_m$, we define the corresponding *Schur polynomial* to be the polynomial extension of the function

$$s_{\boldsymbol{\lambda}}(a_1, \dots, a_m) := \frac{\det(a_j^{\lambda_k})_{j,k=1}^m}{V(a_1, \dots, a_m)}$$

for distinct $a_1, \dots, a_m \in \mathbb{F}$. If consecutive exponents are equal, the Schur polynomial is identically zero.

Maclaurin coefficients

Given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (0, \infty)^m$, where $m \geq 2$:

$$\Lambda_{\boldsymbol{\alpha}}^{(n)}(0^+) = \begin{cases} 0 & \text{if } n \leq m-2 \\ (-1)^{n-m+1}\, a_1 \cdots a_m\, s_{(0,1,\ldots,m-2,n)}(a_1, \ldots, a_m) & \text{if } n \geq m-1 \end{cases}$$

where $a_j = \alpha_j^{-1}$ for $j = 1, \ldots, m$.
The parameter $\boldsymbol{\alpha}$ can be recovered, up to permutation of its entries, from the first $m+1$ non-trivial Maclaurin coefficients,

$$\Lambda_{\boldsymbol{\alpha}}^{(m-1)}(0^+), \ldots, \Lambda_{\boldsymbol{\alpha}}^{(2m-1)}(0^+).$$

Moments

The Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$ has pth moment

$$\begin{aligned}\mu_p &:= \int_{\mathbb{R}} x^p \Lambda_{\boldsymbol{\alpha}}(x)\,\mathrm{d}x \\ &= p!\, s_{(0,1,\ldots,m-2,m-1+p)}(\alpha_1, \ldots, \alpha_m) \quad \text{for all } p \geq 0.\end{aligned}$$

The parameter $\boldsymbol{\alpha}$ can be recovered, up to permutation of its entries, from the first m moments, $\mu_1, \ldots, \mu_m$.

Part 4: Orbital Integrals

Second probabilistic interpretation

Consider the diagonal matrix $D = \mathrm{diag}(a_1, \ldots, a_m)$ and its orbit Ω under unitary conjugation in the space of $m \times m$ positive semi-definite matrices.

If μ is the normalized $U(m)$-invariant measure on Ω then $\Lambda_{\boldsymbol{\alpha}}(x)\,\mathrm{d}x$ is the distribution of any diagonal entry of a random positive semi-definite matrix of arbitrary size distributed according to μ.

Group representation link

Vershik and Kerov, 1982. *"It is worth mentioning that, in works going back to the 30s, Schoenberg, but also Krein and Gantmacher, and later Karlin, have developed the theory of totally positive kernels and matrices. However, a connection with the characters of the unitary group and Weyl's formula was not remarked at that time."*

Representations of the "big" groups

That happened later, with a series of spectacular discoveries: Fourier transforms of Pólya frequency functions were rediscovered as:
Thoma, 1964 irreducible characters of representations of the infinite symmetric group,

and independently
Voiculescu, 1976 irreducible characters of unitary representations of the infinite unitary group $U(\infty)$.

In addition,
Gelfand, Naimark, 1951 classification and explicit expression of spherical/zonal functions associated to $GL(n, \mathbb{C})/U(n)$ pointed out to a similar class of determinantal formulae.

Third probabilistic interpretation

One collects n independent observations $X^{(1)}, \dots, X^{(n)}$ from a p-dimensional Gaussian vector $X \sim N(\mu, \Sigma)$ with mean $\mu \in \mathbb{R}^p$ and covariance matrix $\Sigma \in \mathbb{R}^{p \times p}$.

Detect non-spurious correlations between the components of X. If $\mathbf{X}$ denotes the $n \times p$ matrix whose rows are the vectors $X^{(1)}, \dots, X^{(n)}$, then detecting dependencies between the variables can be performed by computing the associated *sample covariance matrix*, which, up to rescaling, equals $\mathbf{X}^T\mathbf{X}$.

A large entry in $\mathbf{X}^T\mathbf{X}$ indicates a high level of dependence between the corresponding variables and understanding the exact distribution of $\mathbf{X}^T\mathbf{X}$ is critical to assessing whether a large entry arises purely by chance or as the result of a real interaction.

Wishart, 1928, was able to compute the density of such matrices in closed form via changes of variables and the calculation of Jacobians. The resulting distribution is named after him.

When the observations $X^{(1)}, \ldots, X^{(n)}$ have different means μ_1, $\ldots$, μ_n (the *non-central* case), computing the density of $\mathbf{X}^T\mathbf{X}$ involves integrals over the orthogonal group and the result can be expressed in terms of zonal polynomials.

Orbital integrals

The Fourier transforms of Pólya frequency functions can be viewed as characteristic functions of certain unitarily invariant measures defined on the space of infinite Hermitian matrices. The next few slides follow the more recent works of **Olshanski and Vershik, 1996**.

Let $U(n)$ denote the compact group of unitary transformations of $\mathbb{C}^n$, let $\Omega = \{USU^* : U \in U(n)\}$ denote the orbit under unitary conjugation of an $n \times n$ Hermitian matrix $S \in H(n)$, and let μ denote the $U(n)$-invariant probability measure carried by Ω.

The *characteristic function* of μ is

$$f_S : H(n) \to \mathbb{C};\ B \mapsto \int_\Omega e^{\mathrm{i}\,\mathrm{tr}(BM)} \mu(\mathrm{d}M).$$

This function is invariant under unitary conjugation, so that

$$f_S(UBU^*) = f_S(B) \qquad \text{for all } U \in U(n) \text{ and } B \in H(n).$$

Orbital integrals

Hence $f_S(B)$ depends only on the eigenvalues of B and is a symmetric function of these eigenvalues. Furthermore, as a Fourier transform, the function f_S is positive definite.

Consider the inductive limit of such measures and functions defined on the space $H(\infty) = \bigcup_n H(n)$ of Hermitian matrices of arbitrary size. The functions, normalized by the condition $f(0) = 1$, form a convex set and the extremal points of this set are multiplicative, in the sense that

$$f\left(\operatorname{diag}(b_1, b_2, \ldots, b_m)\right) = F(b_1)F(b_2)\cdots F(b_m)$$

for some function F.

This situation occurs precisely when the corresponding unitarily invariant measure μ on the union $H(\infty)$ is *ergodic*: every unitarily invariant Borel subset has measure either 0 or 1.

Olshanski and Vershik main result provides a *bijective correspondence between ergodic, unitarily invariant probability measures on* $H(\infty)$ *and Pólya frequency functions.*

Spherical means

The multiplicative factor F is the Fourier transform of a Pólya frequency function attached to the ergodic measure μ.

Some specific invariant measures provide the building blocks of the class of Pólya frequency functions.

Let $S = \mathrm{diag}(a_1, \dots, a_m)$, where $a_1, \dots, a_m$ are positive, and let $B = E_{11} = \mathrm{diag}(1, 0, 0, \dots, 0)$. The symmetry $f_S(\mathrm{i}xB) = f_B(\mathrm{i}xS)$ implies

$$f_S(\mathrm{i}xE_{11}) = \int_{\Omega'} \exp\Big(-x \sum_{j=1}^{m} a_j |z_j|^2\Big) \sigma(\mathrm{d}z) \qquad (x > 0)$$

where $\Omega' = S^{2m-1} \cong U(m)/U(m-1)$ is the unit sphere in $\mathbb{C}^m$ and σ is the uniform probability measure on the sphere.

The Harish-Chandra–Itzykson–Zuber formula

We claim that, up to an explicit normalisation, the spherical average above is equal to the Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$ at the point x if we suppose that $\boldsymbol{\alpha} = (a_1^{-1}, \ldots, a_m^{-1})$.

If the tuples $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{C}^m$ each have distinct coordinates and $S = \operatorname{diag} \mathbf{a}$ then the orbital integral f_S is given by the Harish-Chandra–Itzykson–Zuber formula:

$$f_S(-\mathrm{i} \operatorname{diag} \mathbf{b}) = \frac{\prod_{j=0}^{m-1} j!}{V(\mathbf{a}) V(\mathbf{b})} \det \begin{pmatrix} e^{b_1 a_1} & e^{b_1 a_2} & \cdots & e^{b_1 a_m} \\ e^{b_2 a_1} & e^{b_2 a_2} & \cdots & e^{b_2 a_m} \\ \vdots & \vdots & \ddots & \vdots \\ e^{b_m a_1} & e^{b_m a_2} & \cdots & e^{b_m a_m} \end{pmatrix}.$$

The orbital-integral representation

If σ denotes the normalized uniform measure on the sphere S^{2m-1} then

$$\Lambda_{\alpha}(x) = \frac{a_1 \cdots a_m}{V(\mathbf{a})} \det \begin{pmatrix} e^{-a_1 x} & e^{-a_2 x} & e^{-a_3 x} & \cdots & e^{-a_m x} \\ 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{m-2} & a_2^{m-2} & a_3^{m-2} & \cdots & a_m^{m-2} \end{pmatrix}$$

$$= \frac{a_1 \cdots a_m}{(m-1)!} x^{m-1} \int_{S^{2m-1}} \exp\Big(-x \sum_{j=1}^{m} a_j |z_j|^2\Big) \sigma(\mathrm{d}z) \quad (x > 0).$$

Spline functions

Let $a_1 < a_2 < \ldots < a_n$ be real numbers. There exists a unique function $f(t) = M_n(a_1, a_2, \ldots, a_n; t)$ of class $C^{(n-3)}$ with the properties:

$$f \geq 0, \quad \mathrm{supp}(f) \subset [a_1, a_n], \int f(t)dt = 1,$$

and the restriction of f to every interval $[a_j, a_{j+1}]$ is a polynomial of degree less than or equal to $n - 2$.

Okounov's formula

The density of the projection μ_S^1 of the orbital measure μ_S (onto the 1×1 entry) is the fundamental spline:

$$\mu_S^1(dt) = M_n(a_1, a_2, \ldots, a_n; t)dt.$$

Part 5: Self-transforms

Transforms of Hirschman–Widder densities

BGKP, 2022. Suppose $m \geq 3$. There exists a subset $\mathcal{N}$ of $(0,\infty)^m$ with Lebesgue measure zero such that

$$p \circ \Lambda_{\boldsymbol{\alpha}} : x \mapsto p\big(\Lambda_{\boldsymbol{\alpha}}(x)\big)$$

is not a Pólya frequency function for any $\boldsymbol{\alpha} \in (0,\infty)^m \setminus \mathcal{N}$ and any real polynomial p that is not a homothety: $p(x) \not\equiv cx$ for any $c > 0$.

Suppose $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (0, \infty)^m$, where $m \geq 2$, is such that the reciprocals $a_1 := \alpha_1^{-1}$, ..., $a_m := \alpha_m^{-1}$ form an arithmetic progression.
Then $c\Lambda_{\boldsymbol{\alpha}}^n$ is a Pólya frequency function for every $c > 0$ and every integer $n \geq 1$.

If, moreover, $\alpha_1 = \alpha_m$ or α_1/α_2 is irrational, then $p \circ \Lambda_{\boldsymbol{\alpha}}$ is not a Pólya frequency function for any other polynomial p.

Rigidity of Pólya frequency functions and similar kernels

BGKP, 2020. There exists a Pólya frequency function M such that M^n is not a Pólya frequency function for any integer $n \geq 2$.

Let $F : [0, \infty) \to [0, \infty)$. The composition map by F preserves the following classes,

1. one-sided Pólya frequency functions,
2. one-sided totally non-negative *measurable* Toeplitz kernels on $\mathbb{R} \times \mathbb{R}$,
3. one-sided Pólya frequency sequences,

if and only if the function F has the following form in each case, where $c > 0$:

1. $F(x) = cx$;
2. $F(x) = cx$, $F(x) = c\mathbf{1}_{x>0}$, or $F(x) = 0$;
3. $F(x) = cx$, or $F(x) = 0$.

Hankel kernels

Of the form $f(x+y)$.

Let $X \subseteq \mathbb{R}$ be an open interval and let $F : [0, \infty) \to \mathbb{R}$. The composition map by F preserves the set of continuous totally non-negtaive Hankel kernels on $X \times X$ if and only if $F(x) = \sum_{k=0}^{\infty} c_k x^k$ on $(0, \infty)$, with $c_k \geq 0$ for all k and $F(0) \geq 0$.

A similar statement holds for preservers of totally positive Hankel kernels on $X \times X$.

Arbitrary totally ordered supports

Let X and Y be infinite, totally ordered sets that admit a totally positive kernel on $X \times Y$. A function $F : (0, \infty) \to (0, \infty)$ preserves by composition the set of totally positive kernels on $X \times Y$ if and only $F(x) = cx$ with $c > 0$.

A similar result holds for the non-constant preservers of totally non-negative kernels on $X \times Y$, and for the preservers of symmetric TP or TN kernels on $X \times X$.

Part 6: Restricted total positivity

Fractional powers

Karlin, 1964. Let $\Omega : \mathbb{R} \to \mathbb{R};\ x \mapsto \mathbf{1}_{x \geq 0}\, xe^{-x}$ be the probability density function for the sum of two independent standard exponential random variables. Given an integer $p \geq 2$ and a scalar $\alpha \geq 0$, the function $\Omega^\alpha : x \mapsto \Omega(x)^\alpha$ is TN_p if $\alpha \in \mathbb{Z}_{\geq 0} \cup (p-2, \infty)$.

The converse

Khare, 2020. Given q, $r \in (0, \infty)$, let $\Omega_{(q,r)}$ be the probability density function for $qX_1 + rX_2$, where X_1 and X_2 are independent standard exponential random variables. Now fix an integer $p \geq 2$ and a real number $\alpha \geq 0$.

1. The function
$$\Omega^{\alpha}_{(q,r)} : \mathbb{R} \to \mathbb{R}; \; x \mapsto \Omega_{(q,r)}(x)^{\alpha}$$
is TN_p if $\alpha \in \mathbb{Z}_{\geq 0} \cup (p-2, \infty)$.
2. If $\alpha \in (0, p-2) \setminus \mathbb{Z}$, then $\Omega^{\alpha}_{(q,r)}$ is not TN_p. More strongly, given arbitrary real numbers $x_1 < \cdots < x_p$ and $y_1 < \cdots < y_p$, there exists $a \in \mathbb{R}$ such that the matrix
$$(\Omega_{(q,r)}(x_j - y_k - a)^{\alpha})_{j,k=1}^{p}$$
is TP if $\alpha > p-2$, TN if $\alpha \in \{0, 1, \ldots, p-2\}$, and has a negative principal minor if $\alpha \in (0, p-2) \setminus \mathbb{Z}$.

Coincidence?

Very similar power spectra appeared in the 1970-ies in rather independent contexts:

Rossi-Vergne. Powers of the Bergman kernel on a tube domain in $\mathbb{C}^n$. Naming the admissible powers the Wallach set.

Berezin. Quantization of classical symmetric domains.

Gindikin. Riesz distributions on symmetric cones.

Fitzgerald-Horn. Entry-wise powers of positive matrices of a fixed size.

Multiply positive kernels: restricted total positivity

Schoenberg, 1955. Let

$$W : \mathbb{R} \to \mathbb{R}; \ x \mapsto \mathbf{1}_{|x| \leq \pi/2} \cos x.$$

For any integer $p \geq 2$, the power W^α is TN_p if and only if $\alpha \geq p - 2$.

Powers of Hirschman-Widder distributions

Karlin, 1968.
If Λ is a HW distribution and $p \geq 2$ is an integer then $\mathcal{B}\{\Lambda\}^{\alpha}$ is the Laplace transform of a TN_p function for all $\alpha \in \mathbb{Z}_{>0} \cup (p-1, \infty)$.

Survey article

With references to the history and several facets of totally positive kernels, specifically addressing Pólya frequency functions.

Belton; Guillot; Khare; Putinar: *Preservers of totally positive kernels and Pólya frequency functions,* Math. Res. Rep. **3** (2022), 35–56.