Spectral deconvolution of unitary invariant models

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(General) spectral deconvolution problem

Spectral measure of $A$:

$$A = A^* \in M_N(\mathbb{C}) \sim \mu_A = \frac{1}{N} \sum_{\lambda \text{ eigvals of } A}^{N} \delta_\lambda$$

Three ingredients:

1. $A \in M_N(\mathbb{C})$, $A = A^*$, the unknown "signal" matrix,
2. $X = (X_1, \ldots, X_r) \in M_N(\mathbb{C})^r$, the unknown, random "noise" matrices, and
3. $W = f(A, X_1, X_1^*, \ldots, X_r, X_r^*)$, the known "data" matrix, with $f \in \mathbb{C}\langle U, V_1, V_1^*, \ldots, V_r, V_r^*\rangle$ self-adjoint polynomial.

Main goal: recover $\mu_A$ from $W$. 
Applications

- Wireless communication:
  \[ W = (X_1 A + X_2)^*(X_1 A + X_2), \]
  with \( X_1, X_2 \) Ginibre matrices (\( A \) matrix of emission powers, \( X_1 \) matrix of random messages, \( X_2 \) noise matrix from environment, \( W \) signal matrix) \( \sim \) General linear model (Ryan, Debbah, 2007).

- Cleaning of covariance matrix:
  \[ W = XAX^* \]
  with \( X \) Ginibre matrix (Ledoit-Péché (2011), Ledoit-Wolf (2013))

- Generalized/structured covariance matrix:
  \[ W = XAX^* \]
  with \( X = BG \ B \succeq 0, \ G \) Ginibre matrix (Buns, Allez, Bouchaud, Potter (2016)). For example \( (B^2)_{ij} = \exp(|i - j|/\tau). \)
Other approaches:

- **Covariance matrices**:
  - Bai and al. (2010): deconvolution using moments.
  - Ledoit and Wolf (2013): "QuEST" algorithm for estimating covariance matrices, numerical inversion of the Marchenko-Pastur equation.

- Bun, Allez, Bouchaud, Potters (2016): Rotationally Invariant Estimator of noisy matrix models, using the spectral deconvolution as an oracle (some spectral deconvolution are achieved using the R/S-transform).


Spectral deconvolution: main assumptions

1. Simplest algebraic combinations: $f = X + Y$ (additive case) or $f = ZYZ^*$ (multiplicative case), and we set $X = Z^*Z$.

2. Control on the spectral distribution of $X$: $\mathbb{E}(d_{Wass,1}(\mu_X, \mu_0)^2) \leq \frac{C}{N^2}$ for some $\mu_0 \in \mathcal{M}_1(\mathbb{R})$.

3. Independence between the eigenbasis of $A$ and $X$: $X \overset{law}{=} UXU^*$ for $U$ Haar unitary.

Main phenomenon: for $N$ large, $\mu_W \simeq \mu_A \boxplus / \boxdot \mu_X$, where $\boxplus, \boxdot$ denotes the free additive/multiplicative convolution of measures.

- as $N$ goes to infinity in law: Voiculescu (1991),
- as $N$ goes to infinity, almost surely: Speicher (1993),
- recently, large deviation principle: Belinschi, Guionnet and Huang (2020).
Free operations : analytic transforms

Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute $\mu_3$ from $\mu_1$ and $\mu_2$?

Cauchy transform of $\mu \in \mathcal{M}_1(\mathbb{R})$: write $\mathbb{C}^\pm = \{ z \in \mathbb{C}, \pm \Im z > 0 \}$,

$$G_\mu = \begin{cases} \mathbb{C}^+ & \mapsto \mathbb{C}^- \\ z & \mapsto \int_\mathbb{R} \frac{1}{z-t} d\mu(t) \end{cases}$$

Stieltjes inversion formula:

$$d\mu(t) = -\frac{1}{\pi} \lim_{y \to 0} \Im G_\mu(t + iy).$$

We also define $F_\mu, h_\mu : \mathbb{C}^+ \to \mathbb{C}^+$ as

$$F_\mu(z) = \frac{1}{G_\mu(z)},$$

$$h_\mu(z) = F_\mu(z) - z.$$
Free operations : the subordination phenomenon

(Belinschi, Bercovici, Biane, Voiculescu)
Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute $\mu_3$ from $\mu_1$ and $\mu_2$?

There exist $\omega_1, \omega_2 : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ such that for $z \in \mathbb{C}^+$,

1. $G_{\mu_3}(z) = G_{\mu_1}(\omega_1(z)) = G_{\mu_2}(\omega_2(z))$ (subordination property)

2. $\omega_1(z) + \omega_2(z) = z + \frac{1}{G_{\mu_3}(z)}$ (free additive relation)

Using 1. and 2., one has access to $\omega_1(z)$ as a fixed point equation/iteration limit

$$\omega_1(z) = K_z(\omega_1(z)) = \lim_{n \rightarrow \infty} K_z^{\circ n}(w),$$

for any $w \in \mathbb{C}^+$, with

$$K_z(w) = h_{\mu_2}(h_{\mu_1}(w) + z) + z.$$
Free operations: the subordination phenomenon

(Belinschi, Bercovici, Biane, Voiculescu)

Suppose that $\mu_3 = \mu_1 \boxtimes \mu_2$. How to compute $\mu_3$ from $\mu_1$ and $\mu_2$?

$(\mu_1, \mu_2 \in \mathcal{M}_1(\mathbb{R}^+))$

There exist $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ such that for $z \in \mathbb{C}^+$,

1. $zG_{\mu_3}(z) = \omega_1(z)G_{\mu_1}(\omega_1(z)) = \omega_2(z)G_{\mu_2}(\omega_2(z))$ (subordination property for $d\tilde{\mu}(t) = td\tilde{\mu}(t)$)

2. $\omega_1(z)\omega_2(z) = \frac{z}{1-z^{-1}F_{\mu}(z)}$ (free multiplicative relation)

Using 1. and 2., one has access to $\omega_1(z)$ as a fixed point equation/iteration limit

$$\omega_1(z) = H_z(\omega_1(z)) = \lim_{n \to \infty} H_z^\circ n(w),$$

for any $w \in \mathbb{C}^+$, with

$$H_z(w) = -\frac{z}{h_{\mu_2}\left(\frac{-z}{h_{\mu_1}(w)}\right)}.$$
Free deconvolution: the subordination method

Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute $\mu_2$ from $\mu_1$ and $\mu_3$?

Recall:

1. $G_{\mu_3}(z) = G_{\mu_1}(\omega_1(z)) = G_{\mu_2}(\omega_2(z))$
2. $\omega_1(z) + \omega_2(z) = z + \frac{1}{G_{\mu_3}(z)}$

Suppose that $\omega_2$ is invertible on some domain $\subset \mathbb{C}^+$:

1. $G_{\mu_3}(\omega_2^{-1}(z)) = G_{\mu_1}(\omega_1(\omega_2^{-1}(z))) = G_{\mu_2}(z)$
2. $\omega_1(\omega_2^{-1}(z)) + z = \omega_2^{-1}(z) + \frac{1}{G_{\mu_3}(\omega_2^{-1}(z))} = \omega_2^{-1}(z) + \frac{1}{G_{\mu_2}(z)}$

Set $\omega_3(z) = \omega_2^{-1}(z), \omega_1'(z) = \omega_1(\omega_2^{-1}(z))$:

1. $G_{\mu_3}(\omega_3(z)) = G_{\mu_1}(\omega_1'(z)) = G_{\mu_2}(z)$
2. $\omega_1'(z) + z = \omega_3(z) + F_{\mu_2}(z)$

Domain of validity?
Free deconvolution : the subordination method

Set $\mathbb{C}_\sigma = \{z \in \mathbb{C}, \Re z > \sigma\}$, and write $\sigma_1^2 = \text{Var}(\mu_1)$.

**Theorem (Arizmendi, Vargas, T.)**

There exist two analytic functions $\omega_1, \omega_3 : \mathbb{C}_{2\sqrt{2}\sigma_1} \to \mathbb{C}^+$ such that for all $z \in \mathbb{C}_{2\sqrt{2}\sigma_1}$,

1. $G_{\mu_2}(z) = G_{\mu_1}(\omega_1(z)) = G_{\mu_2}(\omega_3(z))$ (subordination property).
2. $\omega_1 + z = \omega_3 + F_{\mu_1}(\omega_1(z)) = \omega_3 + F_{\mu_3}(\omega_3(z))$.

Moreover, $\omega_3(z)$ is the unique fixed point of the function $K_z(w) = z - h_{\mu_1}(w + F_{\mu_3}(w) - z)$ in $\mathbb{C}_{3\Im(z)/4}$ and we have

$$\omega_3(z) = \lim_{n \to \infty} K_z^\circ n(w), \ w \in \mathbb{C}_{3\Im(z)/4}.$$ 

[Interesting point : except for the first equality in 1., the latter theorem is not related to $\mu_2$, and still holds without "$\mu_1 \boxplus \mu_2 = \mu_3$".]
Free deconvolution : the subordination method

Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute $\mu_2$ from $\mu_1$ and $\mu_3$?

Deconvolution method:

1. Using the latter theorem, compute $\omega_3$ and $G_{\mu_2} = G_{\mu_3} \circ \omega_3$ on the horizontal line $L = \{x + 2\sqrt{2}\sigma_1 i, x \in \mathbb{R}\}$.

2. Fact: using the Stieltjes inversion formula on $L$ instead of $\mathbb{R}$, we obtain the density $f$ of the classical convolution $\mu_2 \ast C_{2\sqrt{2}\sigma_1}$, where $dC_\lambda(t) = d\text{Cauchy}_\lambda(t) = \frac{\lambda}{\pi(t^2 + \lambda^2)}$.

3. Using classical deconvolution tools, recover $\mu_2$ by doing the deconvolution of $f$ by $C_{2\sqrt{2}\sigma_1}$.

Example of deconvolution tools (regularization needed!):

- regularized Fourier transform,
- Tychonov’s regularization,
- Total Variation minimization procedure.
Back to the spectral deconvolution:

Suppose that $W = X + A$, with $\mu_X \simeq \mu_0$ and $\mu_W \simeq \mu_X \boxplus \mu_1$. How to compute $\mu_A$ from $\mu_0$ and $\mu_W$?

Deconvolution method: set $\sigma = \text{Var}(\mu_0)^{1/2}$.

1. using subordination approach, compute $\omega_3$ and $G = G_{\mu_W} \circ \omega_3$ on the horizontal line $L = \{x + 2\sqrt{2}\sigma, x \in \mathbb{R}\}$ from $\mu_0$ and $\mu_W$ (up to now, $G$ is just a function).

2. Fact: using the Stieltjes inversion formula with $G$ on $L$ instead of $\mathbb{R}$, we obtain a function $\hat{C}$ close to $d\mu_A \ast C_{2\sqrt{2}\sigma}$ on $\mathbb{R}$.

3. Using classical deconvolution tools, achieve the deconvolution of $\hat{C}$ by $C_{2\sqrt{2}\sigma}$ to obtain an estimator $\hat{\mu_A}$ of $\mu_A$.

Main question: how well does $\hat{\mu_A}$ approximate $\mu_A$?
Simulations

**Figure:** Additive case $W = X + A : X \in M_N(\mathbb{C})$ Gaussian Wigner matrix, $A \in M_N(\mathbb{C})$ diagonal with entries iid $\sim N(0, 1)$, $N = 500$. Histogram of $\mu_W$, graph of $\hat{C}$, and comparison between $\hat{\mu}_A$ and $\mu_A$.

**Figure:** Multiplicative case $W = XAX^* : X \in M_N(\mathbb{C})$ Ginibre matrix, $A \in \mathbb{C}$ Wigner with Gaussian entries, $N = 500$. Histogram of $\mu_W$, graph of $\hat{C}$, and comparison between $\hat{\mu}_A$ and $\mu_A$. 
Stability of the free deconvolution: heuristic from the subordination property
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\[ G_{\mu_3}(\mathbb{C}^+) \subset G_{\mu_2}(\mathbb{C}^+). \]
Stability of the free deconvolution: heuristic from the subordination property.
Stability of the free deconvolution: heuristic from the subordination property

Subordination of the deconvolution.
Stability of the free deconvolution: heuristic from the subordination property

Maximum domain of recovery of $G_{\mu_2}$. 

$D$

$\mathbb{R} + i2\sqrt{2}\sigma$

$\mathbb{R}$
Stability of the free deconvolution: heuristic from the subordination property

Extension of $G_{\mu_2}$ from $\mathbb{C}_{2\sqrt{2\text{Var}(\mu_0)}}/D_{\text{max}}$ to $\mathbb{C}^+$. In the case $\mathbb{C}_{2\sqrt{2\text{Var}(\mu_0)}}$, this equivalent to a classical deconvolution.
What about the classical case?

Suppose

- $\mu_1, \mu_2 \in \mathcal{M}_1(\mathbb{R})$,
- we know $\mu_1$ and $\mu_3$ with $\|d\mu_3, d(\mu_1 * \mu_2)\|_{L^2} \leq \epsilon$.

How well can we recover $\mu_2$? Theoretically:

$$\Phi_{\mu_3} \simeq \Phi_{\mu_1} \Phi_{\mu_2},$$

with $\Phi_{\mu}$ the Fourier transform of $\mu$. BUT, in most cases,

$$\Phi_{\mu_1}(t) \xrightarrow{t \to \infty} 0,$$

which prevent the naive method $\widehat{\mu_2} \simeq \Phi^{-1}(\Phi_{\mu_3}/\Phi_{\mu_1})$.

1. Without regularization, nothing can be done: need for hypothesis on $\mu_2$.
2. Smoother $\mu_1$, rougher $\mu_2 \Rightarrow$ Faster exploding of $\Phi_{\mu_3}/\Phi_{\mu_1}$ at $\infty \Rightarrow$ Harder regularization.
The classical case:

\( \hat{\mu}_2 \sim \Phi^{-1}(\Phi_{\mu_3}/\Phi_{\mu_1}) \)

(Butucea, Fan, Lacour and al.)
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\( \hat{\mu}_2 \simeq \Phi^{-1}(1_{[\kappa,\kappa]} \Phi_{\mu_3} / \Phi_{\mu_1}) \)
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\[ \hat{\mu}_2 \sim \Phi^{-1}(1_{[-\kappa, \kappa]} \Phi_{\mu_3}/\Phi_{\mu_1}) \]
The classical case:

(Butucea, Fan, Lacour and al.)

\[ \hat{\mu}_2 \sim \Phi^{-1}(1_{[-K,K]} \Phi \mu_3 / \Phi \mu_1) \]

Then, suppose there exist \( b_i, \gamma_i, s_i > 0 \) with

- \( a \leq |\Phi \mu_1(t)|(1 + t^2)^{\gamma_1/2} \exp(b_1|t|^{s_1}) \leq A, \ a, A > 0 \)
- \( \int_{\mathbb{R}} |\Phi \mu_2(t)|^2(1 + t^2)^{\gamma_2} \exp(2b_2|t|^{s_2}) < L \) with \( L < \infty \),
- \( \|d\mu_3 - d(\mu_1 * \mu_2)\|_{L^2} \leq \epsilon. \)
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Then, an appropriate \( K = K(\epsilon) \) yields \( \|d\hat{\mu}_2 - d\mu_2\|_{L^2}^2 \) bounded by a constant \( C(L, A) \) times

<table>
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The classical case:

(Butucea, Fan, Lacour and al.)

\[ \hat{\mu}_2 \simeq \Phi^{-1}(1_{[-K,K]} \Phi \mu_3 / \Phi \mu_1) \]

Then, suppose there exist \( b_i, \gamma_i, s_i > 0 \) with

- \( a \leq |\Phi \mu_1(t)|(1 + t^2)^{\gamma_1/2} \exp(b_1|t|^{s_1}) \leq A, \ a, A > 0 \)
- \( \int_{\mathbb{R}} |\Phi \mu_2(t)|^2(1 + t^2)^{\gamma_2} \exp(2b_2|t|^{s_2}) < L \) with \( L < \infty \),
- \( \|d\mu_3 - d(\mu_1 * \mu_2)\|_{L^2} \leq \epsilon \).

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Optimal bounds!
The classical/free Cauchy deconvolution:

(Butucea, Fan, Lacour and al.)

If $\mu_1 = C_\lambda$, $\Phi_{\mu_1}(t) = \exp(-\lambda|t|)$.

Hence,

- $a_1 = 1, \gamma_1 = 0, b_1 = \lambda, s_1 = 1$.
- $\int_{\mathbb{R}} |\Phi_{\mu_2}(t)|^2 (1 + t^2)^{\gamma_2} \exp(2b_2|t|^{s_2}) < L$ with $L < \infty$.
- $\|d\mu_3 - d(\mu_1 \ast \mu_2)\|_{L^2} \leq \epsilon$. 
The classical/free Cauchy deconvolution:

(Butucea, Fan, Lacour and al.)

If $\mu_1 = C\lambda$, $\Phi_{\mu_1}(t) = \exp(-\lambda|t|)$.
Hence,

- $a_1 = 1, \gamma_1 = 0, b_1 = \lambda, s_1 = 1$.
- $\int_\mathbb{R} |\Phi_{\mu_2}(t)|^2 (1 + t^2)^{\gamma_2} \exp(2b_2|t|^{s_2}) < L$ with $L < \infty$.
- $\|d\mu_3 - d(\mu_1 * \mu_2)\|_{L^2} \leq \epsilon$.

Then, an appropriate $K = K(\epsilon)$ yields $\|d\hat{\mu}_2 - d\mu_2\|_{L^2}^2$ bounded by a constant $C(L, A)$ times

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The classical classical/free Cauchy deconvolution:

For us: \( \mu_1 = C_2\sqrt{2}\sigma_1 \), \( \Phi_{\mu_1}(t) = \exp(-\lambda|t|) \), \( \mu_2 = \mu_A \) and \( d\mu_3 = \hat{\mathcal{C}} \).

Hence, if

\[
\|\hat{\mathcal{C}} - d(C_2\sqrt{2}\sigma \ast \mu_A)\|_{L^2} \leq \epsilon
\]

an appropriate \( K = K(\epsilon) \) yields that \( \|d\hat{\mu}_A - d\mu_A\|_{L^2}^2 \) is bounded by

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<th>( C(k)(-\log \epsilon)^{-2} )</th>
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<td>( d\mu_A ) can be analytically extended to ( \mathbb{R} \times ] - b_2, b_2 [ )</td>
<td>( \frac{b_2}{\epsilon^{b_2 + 2\sqrt{2}\sigma}} (-\log(\epsilon))^{\xi} )</td>
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There is a version for atomic measures: the condition to ensure well recovery is a minimum separation distance \( t(\lambda) \) between atoms of \( \mu_2 \).

Main goal: to estimate \( \|\hat{\mathcal{C}} - d(C_2\sqrt{2}\sigma \ast \mu_A)\|_{L^2} \).
**Convergence result on the subordination method**

Recall:  \( W = UXU^* + A \),  \( U \) Haar unitary,  \( A, X \in M_N(\mathbb{C}) \) self-adjoint.

**Theorem**

*Suppose that  \( N \geq K_1 \). Then,*

\[
\mathbb{E} \left( \| \hat{C} - d( C_{2\sqrt{2}\tau} \ast \mu_A) \|_{L^2}^2 \right) \leq \frac{K_2}{N^2} + \frac{K_3}{N^3} + \frac{K_4}{N^4},
\]

*with  \( K_1, K_2, K_3, K_4 \) depending explicitly on the first six moments of  \( A \) and  \( X \).*

**Corollary**

*Suppose that  \( d_{\text{Levy}}(\mu_A, \mu) \leq \frac{1}{N} \), with  \( d\mu \) analytically extendable to  \( \mathbb{R} + i \) \( -\tau, \tau[, \tau \in ]0, \infty \), then for  \( N \geq K_1 \),

\[
\mathbb{P}(\| \hat{\mu}_A - \mu_A \|_{L^2} > \delta) \leq \frac{C}{\delta^2} \left( \frac{K_2}{N^2} + \frac{K_3}{N^3} + \frac{K_4}{N^4} \right)^{\frac{\tau}{\tau + 2\sqrt{2}\sigma_1}},
\]

*for some  \( C \) depending on the regularity of  \( \mu \).*
Convergence result on the subordination method: simulations

Figure: Simulation of $\|\hat{C} - d(C_{2\sqrt{2}\sigma_1} \ast \mu_A)\|_{L^2}$ in the additive case for $N$ from 50 to 2000 (with a sampling of size 100 for each size), theoretical bound in Theorem, and ratio of the theoretical bound on the simulated error.
Convergence result on the subordination method: simulations

**Figure**: Simulation of $\| \hat{C} - d(C_\kappa \ast \mu_A) \|_{L^2}$ in the multiplicative case for $N$ from 100 to 2000 (with a sampling of size 100 for each size), theoretical bound in Theorem, and ratio of the theoretical bound on the simulated error.
Matrical subordination in the additive case

(Pastur and Vasilchuk, Kargin)

$$W = UXU^* + A$$

[For $R \in M_N(\mathbb{C})$, $R = R^*$, $G_R(z) = (z - R)^{-1}$, $G_{\mu_R}(z) = \text{tr}((R - z)^{-1})$]

Set $f_A(z) = -\text{tr}(AG_W(z))$ and $f_X(z) = -\text{tr}(XG_W(z))$. Then, set

$$\omega_A(z) = z + \frac{\mathbb{E}(f_X(z))}{\mathbb{E}(G_{\mu_W}(z))}, \quad \omega_X(z) = z + \frac{\mathbb{E}(f_A(z))}{\mathbb{E}(G_{\mu_W}(z))}. \quad (1)$$

The functions $\omega_A, \omega_B$ are called matrical subordination functions.

<table>
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<th>$W = UXU^* + A$</th>
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<td>$\tilde{\omega}_A(z) + \tilde{\omega}<em>X(z) = z + \frac{1}{G</em>{\mu_3}(z)}$</td>
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<td>$\mathbb{E}G_{\mu_W}(z) = G_{\mu_A}(\tilde{\omega}<em>A(z)) + R_A(z)$ $= G</em>{\mu_X}(\tilde{\omega}_X(z)) + R_X(z)$</td>
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1. Multiplicative case ? $(XUAU^*X - z)^{-1} \rightsquigarrow X^{-1}(UAU^* - zX^2)^{-1}X^{-1}$

2. Estimates on $R_A(z), R_X(z)$ ? Estimates on $|G_{\mu_W} - \mathbb{E}G_{\mu_W}|$ ?
Matricial subordination in the additive case

\[ R_A(z) = \frac{\mathbb{E} \left[ (G_{\mu W} - \mathbb{E}G_{\mu W})(\phi - \mathbb{E}\phi) \right] - \mathbb{E} \left[ (f_{X} - \mathbb{E}f_{X})(\psi - \mathbb{E}\psi) \right]}{\mathbb{E}G_{\mu W}}, \]

with \( \phi = \text{tr}(G_A(\omega_A(z))UXU^* \tilde{G}_H) \) and \( \psi = \text{tr}(G_A(\omega_A(z))G_H) \).

- Using Nevanlinna theory and Weingarten formulas gives an upper bound on \( \frac{1}{\mathbb{E}G_{\mu W}} \) in terms of first moments of \( A \) and \( X \).

- Using Poincaré inequality on \( U_N \), matrix Hölder inequalities and Weingarten formulas give an upper bound on \( \text{Var}(G_{\mu W}), \text{Var}(f_{X}), \text{Var}(\phi), \text{Var}(\psi) \) in terms of the first moments of \( X \) and \( A \).

[Reminder of Poincaré inequalities : if \( f : U_N \rightarrow \mathbb{R} \) has mean zero, then \( \int_{U_N} |f|^2 \leq \frac{1}{N} \int_{U_N} \|\nabla f\|^2 \)]

Finally,

\[ |R_A(z)| \leq \frac{K(\Im z)}{N^2}, \]

where \( K \) depends on the first moments of \( A \) and \( X \).
Back to the deconvolution

We suppose here for simplicity \( \mu_X = \mu_0 \). We compare then two systems:

1. the deconvolution system
   \[
   \begin{align*}
   \omega_1(z) + z &= \omega_3(z) + \frac{1}{G_{\mu_X}(\omega_1(z))}, \quad (1) \\
   \omega_1(z) + z &= \omega_3(z) + \frac{1}{G_W(\omega_3(z))}. \quad (2)
   \end{align*}
   \]

2. the matricial subordination system of \( W = UXU^* + A \) at \( \omega_3(z) \):
   \[
   \begin{align*}
   \epsilon_X := R_X(\omega_3(z)), \quad \epsilon_A := R_A(\omega_3(z)).
   \end{align*}
   \]
   \[
   \begin{align*}
   \omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_A}(\omega_A(\omega_3)) + \epsilon_A}, \quad (a) \\
   \omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_X}(\omega_X(\omega_3)) + \epsilon_X} = \omega_3 + \frac{1}{\mathbb{E} G_W(\omega_3)}. \quad (b)
   \end{align*}
   \]

Successively deduce:

1. \( \omega_X(\omega_3) \simeq \omega_1 : (1), (2), (b) \) and the invertibility domain of \( 1/G_\mu \),
2. \( \omega_A(\omega_3) \simeq z : \omega_X \simeq \omega_1, (2) \) and \( (b) \),
3. \( G_{\mu_A}(z) \simeq G_{\mu_A}(\omega_A(\omega_3)) \simeq \mathbb{E} G_W(\omega_3) \simeq G_W(\omega_3) : \omega_A(\omega_3) \simeq z, \) then \( (a) \) and \( (b) \), and finally Poincaré inequalities.
Back to the deconvolution

We suppose here for simplicity $\mu_X = \mu_0$. We compare then two systems:

1. the deconvolution system

$$\begin{cases} \omega_1(z) + z = \omega_3(z) + \frac{1}{G_{\mu_X}(\omega_1(z))}, \quad (1) \\ \omega_1(z) + z = \omega_3(z) + \frac{1}{G_W(\omega_3(z))}. \quad (2) \end{cases}$$

2. the matricial subordination system of $W = UXU^* + A$ at $\omega_3(z)$:

$$\begin{cases} \epsilon_X := R_X(\omega_3(z)), \; \epsilon_A := R_A(\omega_3(z)). \\ \omega_X(\omega_3) + \omega_A(\omega_3) = \omega_3 + \frac{1}{G_{\mu_A}(\omega_A(\omega_3)) + \epsilon_A}, \quad (a) \\ \omega_X(\omega_3) + \omega_A(\omega_3) = \omega_3 + \frac{1}{G_{\mu_X}(\omega_X(\omega_3)) + \epsilon_X} = \omega_3 + \frac{1}{\mathbb{E}G_W(\omega_3)}. \quad (b) \end{cases}$$

Successively deduce:

1. $\omega_X(\omega_3) \simeq \omega_1 : (1), (2), (b)$ and the invertibility domain of $1/G_{\mu}$,
2. $\omega_A(\omega_3) \simeq z : \omega_X \simeq \omega_1$, (2) and (b),
3. $G_{\mu_A}(z) \simeq G_{\mu_A}(\omega_A(\omega_3)) \simeq \mathbb{E}G_W(\omega_3) \simeq G_W(\omega_3) : \omega_A(\omega_3) \simeq z$, then (a) and (b), and finally Poincaré inequalities.
Back to the deconvolution

We suppose here for simplicity $\mu_X = \mu_0$. We compare then two systems:

1. the deconvolution system

\[
\begin{aligned}
\omega_1(z) + z &= \omega_3(z) + \frac{1}{G_{\mu_X}(\omega_1(z))}, \\
\omega_1(z) + z &= \omega_3(z) + \frac{1}{G_W(\omega_3(z))}.
\end{aligned}
\]

2. the matricial subordination system of $W = UXU^* + A$ at $\omega_3(z)$:

\[
\begin{aligned}
\epsilon_X &:= R_X(\omega_3(z)), \quad \epsilon_A := R_A(\omega_3(z)). \\
\omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_A}(\omega_A(\omega_3)) + \epsilon_A}, \quad (a) \\
\omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_X}(\omega_X(\omega_3)) + \epsilon_X} = \omega_3 + \frac{1}{\mathbb{E}G_W(\omega_3)}. \quad (b)
\end{aligned}
\]

Successively deduce:

1. $\omega_X(\omega_3) \simeq \omega_1 : (1), (2), (b)$ and the invertibility domain of $1/G_\mu$,

2. $\omega_A(\omega_3) \simeq z : \omega_X \simeq \omega_1, (2)$ and $(b)$,

3. $G_{\mu_A}(z) \simeq G_{\mu_A}(\omega_A(\omega_3)) \simeq \mathbb{E}G_W(\omega_3) \simeq G_W(\omega_3) : \omega_A(\omega_3) \simeq z$, then $(a)$ and $(b)$, and finally Poincaré inequalities.
Back to the deconvolution

We suppose here for simplicity $\mu_X = \mu_0$. We compare then two systems:

1. the deconvolution system

\[
\begin{cases}
\omega_1(z) + z = \omega_3(z) + \frac{1}{G_{\mu_X}(\omega_1(z))}, \\
\omega_1(z) + z = \omega_3(z) + \frac{1}{G_W(\omega_3(z))}.
\end{cases}
\]

2. the matricial subordination system of $W = UXU^* + A$ at $\omega_3(z)$:

\[
\begin{cases}
\omega_X(\omega_3) + \omega_A(\omega_3) = \omega_3 + \frac{1}{G_{\mu_A}(\omega_A(\omega_3))} + \epsilon_A, \\
\omega_X(\omega_3) + \omega_A(\omega_3) = \omega_3 + \frac{1}{G_{\mu_X}(\omega_X(\omega_3))} + \epsilon_X = \omega_3 + \frac{1}{\mathbb{E}G_W(\omega_3)}.
\end{cases}
\]

Successively deduce:

1. $\omega_X(\omega_3) \simeq \omega_1$: (1), (2), (b) and the invertibility domain of $1/G_{\mu}$,

2. $\omega_A(\omega_3) \simeq z$: $\omega_X \simeq \omega_1$, (2) and (b),

3. $G_{\mu_A}(z) \simeq G_{\mu_A}(\omega_A(\omega_3)) \simeq \mathbb{E}G_W(\omega_3) \simeq G_W(\omega_3)$: $\omega_A(\omega_3) \simeq z$, then (a) and (b), and finally Poincaré inequalities.
Back to the deconvolution

We suppose here for simplicity $\mu_X = \mu_0$. We compare then two systems:

1. the deconvolution system

$$
\begin{align*}
\omega_1(z) + z &= \omega_3(z) + \frac{1}{G_{\mu_X}(\omega_1(z))} , \quad (1) \\
\omega_1(z) + z &= \omega_3(z) + \frac{1}{G_W(\omega_3(z))} . \quad (2)
\end{align*}
$$

2. the matricial subordination system of $W = UXU^* + A$ at $\omega_3(z)$:

$$
\begin{align*}
\epsilon_X &:= R_X(\omega_3(z), \epsilon_A := R_A(\omega_3(z)). \\
\begin{cases}
\omega_X(\omega_3) + \omega_A(\omega_3) = \omega_3 + \frac{1}{G_{\mu_A}(\omega_A(\omega_3)) + \epsilon_A} , \quad (a) \\
\omega_X(\omega_3) + \omega_A(\omega_3) = \omega_3 + \frac{1}{G_{\mu_X}(\omega_X(\omega_3)) + \epsilon_X} = \omega_3 + \frac{1}{\mathbb{E}G_W(\omega_3)} . \quad (b)
\end{cases}
\end{align*}
$$

Successively deduce:

1. $\omega_X(\omega_3) \simeq \omega_1$ : (1), (2), (b) and the invertibility domain of $1/G_{\mu}$,
2. $\omega_A(\omega_3) \simeq z : \omega_X \simeq \omega_1$, (2) and (b),
3. $G_{\mu_A}(z) \simeq G_{\mu_A}(\omega_A(\omega_3)) \simeq \mathbb{E}G_W(\omega_3) \simeq G_W(\omega_3) : \omega_A(\omega_3) \simeq z$, then (a) and (b), and finally Poincaré inequalities.
Back to the deconvolution

We suppose here for simplicity $\mu_X = \mu_0$. We compare then two systems:

1. the deconvolution system

$$\begin{align*}
\omega_1(z) + z &= \omega_3(z) + \frac{1}{G_{\mu_X}(\omega_1(z))}, \\
\omega_1(z) + z &= \omega_3(z) + \frac{1}{G_W(\omega_3(z))}.
\end{align*}$$

2. the matricial subordination system of $W = UXU^* + A$ at $\omega_3(z)$:

$$\begin{align*}
\epsilon_X := R_X(\omega_3(z)), \\
\epsilon_A := R_A(\omega_3(z)).
\end{align*}$$

$$\begin{align*}
\omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_A}(\omega_A(\omega_3)) + \epsilon_A}, \\
\omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_X}(\omega_X(\omega_3)) + \epsilon_X} = \omega_3 + \frac{1}{\mathbb{E}G_W(\omega_3)}. \\
\end{align*}$$

Successively deduce:

1. $\omega_X \simeq \omega_1$: (1), (2), (b) and the invertibility domain of $1/G_{\mu}$,

2. $\omega_A(\omega_3) \simeq z : \omega_X \simeq \omega_1$, (2) and (b),

3. $G_{\mu_A}(z) \simeq G_{\mu_A}(\omega_A(\omega_3)) \simeq \mathbb{E}G_W(\omega_3) \simeq G_W(\omega_3) : \omega_A(\omega_3) \simeq z$, then (a) and (b), and finally Poincaré inequalities.
Back to the deconvolution

We suppose here for simplicity $\mu_X = \mu_0$. We compare then two systems:

1. the deconvolution system

$$
\begin{align*}
\omega_1(z) + z &= \omega_3(z) + \frac{1}{G_{\mu_X}(\omega_1(z))}, \\
\omega_1(z) + z &= \omega_3(z) + \frac{1}{G_W(\omega_3(z))}.
\end{align*}
$$

2. the matricial subordination system of $W = UXU^* + A$ at $\omega_3(z)$:

$$
\begin{align*}
\epsilon_X := R_X(\omega_3(z)), \epsilon_A := R_A(\omega_3(z)). \\
\omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_A}(\omega_A(\omega_3))} + \epsilon_A, \\
\omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_X}(\omega_X(\omega_3))} + \epsilon_X = \omega_3 + \frac{1}{\mathbb{E} G_W(\omega_3)}. 
\end{align*}
$$

Successively deduce:

1. $\omega_X \simeq \omega_1$: (1), (2), (b) and the invertibility domain of $1/G_{\mu}$,
2. $\omega_A(\omega_3) \simeq z$: $\omega_X \simeq \omega_1$, (2) and (b),
3. $G_{\mu_A}(z) \simeq G_{\mu_A}(\omega_A(\omega_3)) \simeq \mathbb{E} G_W(\omega_3) \simeq G_W(\omega_3): \omega_A(\omega_3) \simeq z$, then (a) and (b), and finally Poincaré inequalities.
Back to the deconvolution

We suppose here for simplicity $\mu_X = \mu_0$. We compare then two systems:

1. the deconvolution system
   \[
   \begin{align*}
   \omega_1(z) + z &= \omega_3(z) + \frac{1}{G_{\mu_X}(\omega_1(z))}, \quad (1) \\
   \omega_1(z) + z &= \omega_3(z) + \frac{1}{G_W(\omega_3(z))}. \quad (2)
   \end{align*}
   \]

2. the matricial subordination system of $W = UXU^* + A$ at $\omega_3(z)$:
   \[
   \begin{align*}
   \epsilon_X &= R_X(\omega_3(z)), \quad \epsilon_A := R_A(\omega_3(z)). \\
   \omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_A}(\omega_A(\omega_3)) + \epsilon_A}, \quad (a) \\
   \omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_X}(\omega_X(\omega_3)) + \epsilon_X} = \omega_3 + \frac{1}{\mathbb{E}G_W(\omega_3)}. \quad (b)
   \end{align*}
   \]

Successively deduce:

1. $\omega_X \simeq \omega_1 : (1), (2), (b)$ and the invertibility domain of $1/G_{\mu}$,
2. $\omega_A(\omega_3) \simeq z : \omega_X \simeq \omega_1$, (2) and (b),
3. $G_{\mu_A}(z) \simeq G_{\mu_A}(\omega_A(\omega_3)) \simeq \mathbb{E}G_W(\omega_3) \simeq G_W(\omega_3) : \omega_A(\omega_3) \simeq z$, then (a) and (b), and finally Poincaré inequalities.
Back to the deconvolution

We suppose here for simplicity $\mu_X = \mu_0$. We compare then two systems:

1. the deconvolution system

\[
\begin{align*}
\omega_1(z) + z &= \omega_3(z) + \frac{1}{G_{\mu_X}(\omega_1(z))}, \\
\omega_1(z) + z &= \omega_3(z) + \frac{1}{G_W(\omega_3(z))}.
\end{align*}
\] (1) (2)

2. the matricial subordination system of $W = UXU^* + A$ at $\omega_3(z)$:

\[
\begin{align*}
\epsilon_X := R_X(\omega_3(z)), \quad \epsilon_A := R_A(\omega_3(z))
\end{align*}
\]

\[
\begin{align*}
\omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_A}(\omega_A(\omega_3)) + \epsilon_A}, \\
\omega_X(\omega_3) + \omega_A(\omega_3) &= \omega_3 + \frac{1}{G_{\mu_X}(\omega_X(\omega_3)) + \epsilon_X} = \omega_3 + \frac{1}{E G_W(\omega_3)}.
\end{align*}
\] (a) (b)

Successively deduce:

1. $\omega_X \simeq \omega_1$: (1), (2), (b) and the invertibility domain of $1/G_\mu$,

2. $\omega_A(\omega_3) \simeq z$: $\omega_X \simeq \omega_1$, (2) and (b),

3. $G_{\mu_A}(z) \simeq G_{\mu_A}(\omega_A(\omega_3)) \sim E G_W(\omega_3) \simeq G_W(\omega_3): \omega_A(\omega_3) \simeq z$, then (a) and (b), and finally Poincaré inequalities.
Back to the deconvolution

We suppose here for simplicity $\mu_X = \mu_0$. We compare then two systems :

1. the deconvolution system

$$\omega_1(z) + z = \omega_3(z) + \frac{1}{G_{\mu X}(\omega_1(z))}, \quad (1)$$
$$\omega_1(z) + z = \omega_3(z) + \frac{1}{G_W(\omega_3(z))}. \quad (2)$$

2. the matricial subordination system of $W = UXU^* + A$ at $\omega_3(z)$ :

$$\epsilon_X := R_X(\omega_3(z)), \epsilon_A := R_A(\omega_3(z)).$$

$$\omega_3(\omega_3) + \omega_A(\omega_3) = \omega_3 + \frac{1}{G_{\mu A}(\omega_A(\omega_3)) + \epsilon_A}, \quad (a)$$
$$\omega_X(\omega_3) + \omega_A(\omega_3) = \omega_3 + \frac{1}{G_{\mu X}(\omega_X(\omega_3)) + \epsilon_X} = \omega_3 + \frac{1}{E G_W(\omega_3)}. \quad (b)$$

Successively deduce :

1. $\omega_X \simeq \omega_1 : (1), (2), (b)$ and the invertibility domain of $1/G_{\mu}$,
2. $\omega_A(\omega_3) \simeq z : \omega_X \simeq \omega_1$, (2) and (b),
3. $G_{\mu A}(z) \simeq G_{\mu A}(\omega_A(\omega_3)) \simeq E G_W(\omega_3) \simeq G_W(\omega_3) : \omega_A(\omega_3) \simeq z$, then (a) and (b), and finally Poincaré inequalities.
**$L^2$-estimates**

We finally get for $\mathfrak{F}(z) \geq 2\sqrt{2\text{Var}(\mu_0)}$ 

$$|G_{\mu_A}(z) - G_W(\omega_3)| \leq \frac{C_1(\mathfrak{F}z)}{|z|N^2} + \frac{C_2(\mathfrak{F}z)|G_W(\omega_3) - \mathbb{E}G_W(\omega_3)|}{|z|}.$$ 

Recall 

- $\mathbb{E}(|G_W(\omega_3(z)) - \mathbb{E}G_W(\omega_3(z))|^2) \leq \frac{C(\mathfrak{F}z)}{N^2}$, 
- $t \mapsto \frac{1}{|t+i\eta|} \in L^2(\mathbb{R})$. 

Thus 

$$\mathbb{E}\left(\int_{\mathbb{R}} \left| \mathfrak{F}G_W(\omega_3(t + 2\sqrt{2\text{Var}(\mu_0)i})) - \mathfrak{F}G_{\mu_A}(t + 2\sqrt{2\text{Var}(\mu_0)i}) \right|^2 dt \right) \leq \frac{K_2}{N^2} + \frac{K_3}{N^3} + \frac{K_4}{N^4}. $$
We finally get for $\mathfrak{F}(z) \geq 2\sqrt{2 \text{Var}(\mu_0)}$

$$|G_{\mu_A}(z) - G_W(\omega_3)| \leq \frac{C_1(\mathfrak{F}z)}{|z|N^2} + \frac{C_2(\mathfrak{F}z)|G_W(\omega_3) - \mathbb{E}G_W(\omega_3)|}{|z|}.$$  

Recall

- $\mathbb{E}(|G_W(\omega_3(z)) - \mathbb{E}G_W(\omega_3(z))|^2) \leq \frac{C(\mathfrak{F}z)}{N^2}$,
- $t \mapsto \frac{1}{|t+i\eta|} \in L^2(\mathbb{R})$.

Thus,

$$\mathbb{E}\left(\|\hat{C} - d(\mu_A \ast C_2\sqrt{2\text{Var}(\mu_0)}\|^2_{L^2})\right) \leq \frac{K_2}{N^2} + \frac{K_3}{N^3} + \frac{K_4}{N^4}.$$
THANK YOU!