Spectral deconvolution of unitary invariant models

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(General) spectral deconvolution problem

Spectral measure of A :

$$A = A^* \in M_N(\mathbb{C}) \rightsquigarrow \mu_A = \frac{1}{N} \sum_{\lambda \text{ eigvals of } A}^{N} \delta_\lambda$$

Three ingredients :

1. $A \in M_N(\mathbb{C})$, $A = A^*$, the *unknown* "signal" matrix,
2. $\mathbf{X} = (X_1, \ldots, X_r) \in M_N(\mathbb{C})^r$, the *unknown, random* "noise" matrices, and
3. $W = f(A, X_1, X_1^*, \ldots, X_r, X_r^*)$, the *known* "data" matrix, with $f \in \mathbb{C}\langle U, V_1, V_1^*, \ldots, V_r, V_r^* \rangle$ self-adjoint polynomial.

Main goal : recover μ_A from W.

Applications

- Wireless communication :

$$W = (X_1 A + X_2)^*(X_1 A + X_2),$$

with X_1, X_2 Ginibre matrices (A matrix of emission powers, X_1 matrix of random messages, X_2 noise matrix from environment, W signal matrix) $\rightsquigarrow$ General linear model (Ryan, Debbah, 2007).

- Cleaning of covariance matrix :

$$W = XAX^*$$

with X Ginibre matrix (Ledoit-Péché (2011), Ledoit-Wolf (2013))

- Generalized/structured covariance matrix :

$$W = XAX^*$$

with $X = BG$ $B \geq 0$, G Ginibre matrix (Buns, Allez, Bouchaud, Potter (2016)). For example $(B^2)_{ij} = \exp(|i - j|/\tau)$.

Other approaches:

- Covariance matrices :
  - El Karoui (2008) : first results on the subject.
  - Rao (2008) : atomic case.
  - Bai and al. (2010) : deconvolution using moments.
  - Ledoit and Wolf (2013) : "QuEST" algorithm for estimating covariance matrices, numerical inversion of the Marchenko-Pastur equation.
- Bun, Allez, Bouchaud, Potters (2016) : Rotationally Invariant Estimator of noisy matrix models, using the spectral deconvolution as an oracle (some spectral deconvolution are achieved using the R/S-transform).
- Hayase (2020) : use of Cauchy distribution.
- Maïda et al. (preprint, 2020) : study of the backward Fokker-Planck equation.

Spectral deconvolution : main assumptions

1. Simplest algebraic combinations: $f = X + Y$ (*additive case*) or $f = ZYZ^*$ (*multiplicative case*), and we set $X = Z^*Z$.
2. Control on the spectral distribution of X : $\mathbb{E}(d_{Wass,1}(\mu_X, \mu_0)^2) \leq \frac{C}{N^2}$ for some $\mu_0 \in \mathcal{M}_1(\mathbb{R})$.
3. Independence between the eigenbasis of A and X : $X \underset{law}{=} UXU^*$ for U Haar unitary.

Main phenomenon : for N large, $\mu_W \simeq \mu_A \boxplus / \boxtimes \mu_X$, where $\boxplus, \boxtimes$ denotes the free additive/multiplicative convolution of measures.

- as N goes to infinity in law : Voiculescu (1991),
- as N goes to infinity, almost surely : Speicher (1993),
- concentration results for fixed N : Kargin (2015), Bao, Erdösz and Schnelli (2017), Meckes and Meckes (2013).
- recently, large deviation principle : Belinschi, Guionnet and Huang (2020).

Free operations : analytic transforms

Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute μ_3 from μ_1 and μ_2 ?

Cauchy transform of $\mu \in \mathcal{M}_1(\mathbb{R})$: write $\mathbb{C}^\pm = \{z \in \mathbb{C}, \pm \Im z > 0\}$,

$$G_\mu = \begin{cases} \mathbb{C}^+ & \longrightarrow & \mathbb{C}^- \\ z & \mapsto & \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t) \end{cases}$$

Stieltjes inversion formula :

$$d\mu(t) = -\frac{1}{\pi} \lim_{y \to 0} \Im G_\mu(t + iy).$$

We also define $F_\mu, h_\mu : \mathbb{C}^+ \to \mathbb{C}^+$ as

$$F_\mu(z) = \frac{1}{G_\mu(z)},$$

$$h_\mu(z) = F_\mu(z) - z.$$

Free operations : the subordination phenomenon

(Belinschi, Bercovici, Biane, Voiculescu)
Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute μ_3 from μ_1 and μ_2 ?

There exist $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ such that for $z \in \mathbb{C}^+$,

1. $G_{\mu_3}(z) = G_{\mu_1}(\omega_1(z)) = G_{\mu_2}(\omega_2(z))$ (*subordination property*)
2. $\omega_1(z) + \omega_2(z) = z + \frac{1}{G_{\mu_3}(z)}$ (*free additive relation*)

Using 1. and 2., one has access to $\omega_1(z)$ as a fixed point equation/iteration limit

$$\omega_1(z) = K_z(\omega_1(z)) = \lim_{n\to\infty} K_z^{\circ n}(w),$$

for any $w \in \mathbb{C}^+$, with

$$K_z(w) = h_{\mu_2}(h_{\mu_1}(w) + z) + z.$$

Free operations : the subordination phenomenon

(Belinschi, Bercovici, Biane, Voiculescu)
Suppose that $\mu_3 = \mu_1 \boxtimes \mu_2$. How to compute μ_3 from μ_1 and μ_2 ?
$(\mu_1, \mu_2 \in \mathcal{M}_1(\mathbb{R}^+))$

There exist $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ such that for $z \in \mathbb{C}^+$,

1. $zG_{\mu_3}(z) = \omega_1(z)G_{\mu_1}(\omega_1(z)) = \omega_2(z)G_{\mu_2}(\omega_2(z))$ (*subordination property* for $d\tilde{\mu}(t) = td\tilde{\mu}(t)$)
2. $\omega_1(z)\omega_2(z) = \frac{z}{1-z^{-1}F_\mu(z)}$ (*free multiplicative relation*)

Using 1. and 2., one has access to $\omega_1(z)$ as a fixed point equation/iteration limit

$$\omega_1(z) = H_z(\omega_1(z)) = \lim_{n\to\infty} H_z^{\circ n}(w),$$

for any $w \in \mathbb{C}^+$, with

$$H_z(w) = -\frac{z}{h_{\mu_2}\left(\frac{-z}{h_{\mu_1}(w)}\right)}.$$

Free deconvolution : the subordination method

Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute μ_2 from μ_1 and μ_3 ?

Recall :

1. $G_{\mu_3}(z) = G_{\mu_1}(\omega_1(z)) = G_{\mu_2}(\omega_2(z))$
2. $\omega_1(z) + \omega_2(z) = z + \frac{1}{G_{\mu_3}(z)}$

Suppose that ω_2 is invertible on some domain ? $\subset \mathbb{C}^+$:

1. $G_{\mu_3}(\omega_2^{-1}(z)) = G_{\mu_1}(\omega_1(\omega_2^{-1}(z))) = G_{\mu_2}(z)$
2. $\omega_1(\omega_2^{-1}(z)) + z = \omega_2^{-1}(z) + \frac{1}{G_{\mu_3}(\omega_2^{-1}(z))} = \omega_2^{-1}(z) + \frac{1}{G_{\mu_2}(z)}$

Set $\omega_3(z) = \omega_2^{-1}(z), \omega_{1'}(z) = \omega_1(\omega_2^{-1}(z))$:

1. $G_{\mu_3}(\omega_3(z)) = G_{\mu_1}(\omega_{1'}(z)) = G_{\mu_2}(z)$
2. $\omega_{1'}(z) + z = \omega_3(z) + F_{\mu_2}(z)$

Domain of validity ?

Free deconvolution : the subordination method

Set $\mathbb{C}_\sigma = \{z \in \mathbb{C}, \Im z > \sigma\}$, and write $\sigma_1^2 = \mathrm{Var}(\mu_1)$.

Theorem (Arizmendi, Vargas, T.)

There exist two analytic functions $\omega_1, \omega_3 : \mathbb{C}_{2\sqrt{2}\sigma_1} \to \mathbb{C}^+$ *such that for all* $z \in \mathbb{C}_{2\sqrt{2}\sigma_1}$*,*

1. $G_{\mu_2}(z) = G_{\mu_1}(\omega_1(z)) = G_{\mu_2}(\omega_3(z))$ *(subordination property).*
2. $\omega_1 + z = \omega_3 + F_{\mu_1}(\omega_1(z)) = \omega_3 + F_{\mu_3}(\omega_3(z))$.

Moreover, $\omega_3(z)$ *is the unique fixed point of the function* $K_z(w) = z - h_{\mu_1}(w + F_{\mu_3}(w) - z)$ *in* $\mathbb{C}_{3\Im(z)/4}$ *and we have*

$$\omega_3(z) = \lim K_z^{\circ n}(w),\ w \in \mathbb{C}_{3\Im(z)/4}.$$

[Interesting point : except for the first equality in 1., the latter theorem is not related to μ_2, and still holds without "$\mu_1 \boxplus \mu_2 = \mu_3$"]

Free deconvolution : the subordination method

Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute μ_2 from μ_1 and μ_3 ?

Deconvolution method :

1. using the latter theorem, compute ω_3 and $G_{\mu_2} = G_{\mu_3} \circ \omega_3$ on the horizontal line $L = \{x + 2\sqrt{2}\sigma_1 i, x \in \mathbb{R}\}$.
2. Fact : using the Stieltjes inversion formula on L instead of $\mathbb{R}$, we obtain the density f of the classical convolution $\mu_2 * \mathcal{C}_{2\sqrt{2}\sigma_1}$, where $d\mathcal{C}_\lambda(t) = dCauchy_\lambda(t) = \frac{\lambda}{\pi(t^2+\lambda^2)}$.
3. Using classical deconvolution tools, recover μ_2 by doing the deconvolution of f by $\mathcal{C}_{2\sqrt{2}\sigma_1}$.

Exemple of deconvolution tools (regularization needed !):

- regularized Fourier transform,
- Tychonov's regularization,
- Total Variation minimization procedure.

Back to the spectral deconvolution :

Suppose that $W = X + A$, with $\mu_X \simeq \mu_0$ and $\mu_W \simeq \mu_X \boxplus \mu_1$. How to compute μ_A from μ_0 and μ_W ?

Deconvolution method : set $\sigma = Var(\mu_0)^{1/2}$.

1. using subordination approach, compute ω_3 and $\mathcal{G} = G_{\mu_W} \circ \omega_3$ on the horizontal line $L = \{x + 2\sqrt{2}\sigma, x \in \mathbb{R}\}$ from μ_0 and μ_W (up to now, $\mathcal{G}$ is just a function).
2. Fact : using the Stieltjes inversion formula with $\mathcal{G}$ on L instead of $\mathbb{R}$, we obtain a function $\widehat{C}$ close to $d\mu_A * \mathcal{C}_{2\sqrt{2}\sigma}$ on $\mathbb{R}$.
3. Using classical deconvolution tools, achieve the deconvolution of $\widehat{C}$ by $\mathcal{C}_{2\sqrt{2}\sigma}$ to obtain an estimator $\widehat{\mu_A}$ of μ_A.

Main question : how well does $\widehat{\mu_A}$ approximate μ_A ?

Simulations

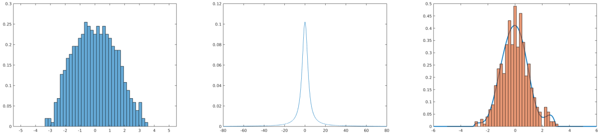

Figure: Additive case $W = X + A$: $X \in M_N(\mathbb{C})$ Gaussian Wigner matrix, $A \in M_N(\mathbb{C})$ diagonal with entries iid $\sim N(0,1)$, $N = 500$.
Histogram of μ_W, graph of $\widehat{C}$, and comparison between $\widehat{\mu_A}$ and μ_A.

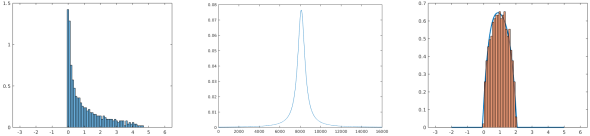

Figure: Multiplicative case $W = XAX^*$: $X \in M_N(\mathbb{C})$ Ginibre matrix, $A \in \mathbb{C}$ Wigner with Gaussian entries, $N = 500$.
Histogram of μ_W, graph of $\widehat{C}$, and comparison between $\widehat{\mu_A}$ and μ_A.

Stability of the free deconvolution : heuristic from the subordination property

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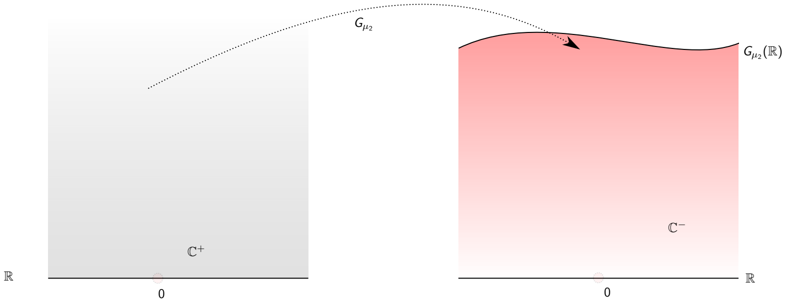
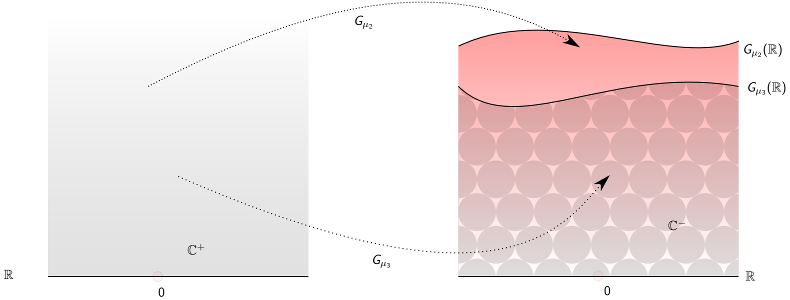
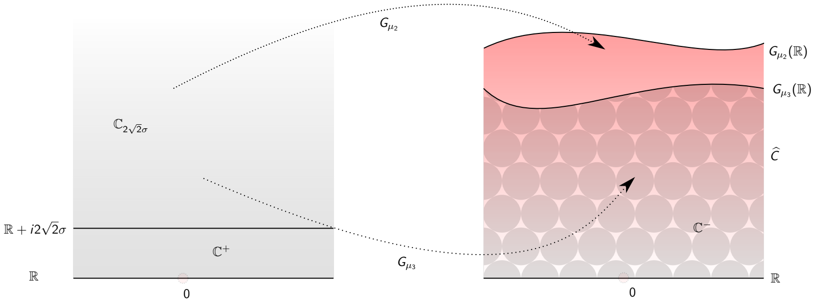


Stability of the free deconvolution : heuristic from the subordination property

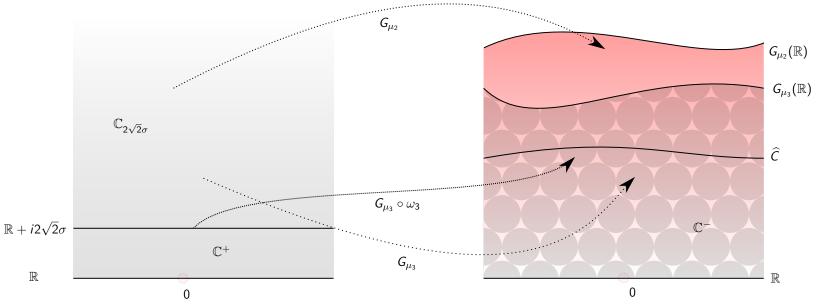


$$G_{\mu_3}(\mathbb{C}^+) \subset G_{\mu_2}(\mathbb{C}^+).$$

Stability of the free deconvolution : heuristic from the subordination property.

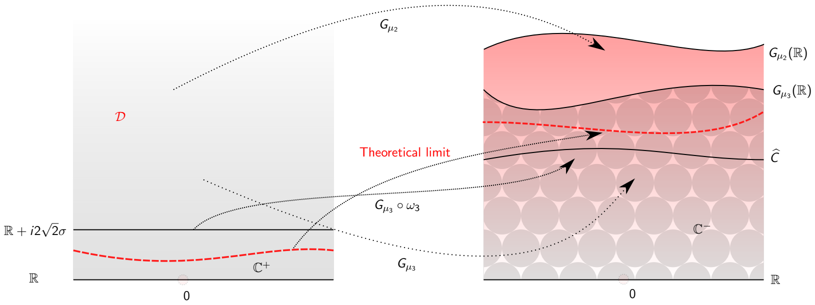


Stability of the free deconvolution : heuristic from the subordination property



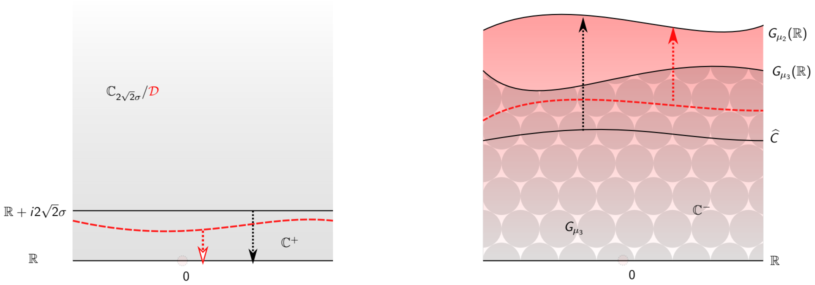
Subordination of the deconvolution.

Stability of the free deconvolution : heuristic from the subordination property



Maximum domain of recovery of G_{μ_2}.

Stability of the free deconvolution : heuristic from the subordination property



Extension of G_{μ_2} from $\mathbb{C}_{2\sqrt{2Var(\mu_0)}}/\mathcal{D}_{\max}$ to $\mathbb{C}^+$.

In the case $\mathbb{C}_{2\sqrt{2Var(\mu_0)}}$, this equivalent to a classical deconvolution.

What about the classical case ?

Suppose

- $\mu_1, \mu_2 \in \mathcal{M}_1(\mathbb{R})$,
- we know μ_1 and μ_3 with $\|d\mu_3, d(\mu_1 * \mu_2)\|_{L^2} \leq \epsilon$.

How well can we recover μ_2 ? Theoretically :

$$\Phi_{\mu_3} \simeq \Phi_{\mu_1}\Phi_{\mu_2},$$

with Φ_μ the Fourier transform of μ. BUT, in most cases,

$$\Phi_{\mu_1}(t) \xrightarrow[t\to\infty]{} 0,$$

which prevent the naive method $\widehat{\mu_2} \simeq \Phi^{-1}(\Phi_{\mu_3}/\Phi_{\mu_1})$.

1. Without regularization, nothing can be done : need for hypothesis on μ_2.
2. Smoother μ_1, rougher μ_2 $\Rightarrow$ Faster exploding of $\Phi_{\mu_3}/\Phi_{\mu_1}$ at ∞ $\Rightarrow$ Harder regularization.

The classical case :

(Butucea, Fan, Lacour and al.)

$$\widehat{\mu_2} \simeq \Phi^{-1}(\Phi_{\mu_3}/\Phi_{\mu_1})$$

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$$\widehat{\mu_2} \simeq \Phi^{-1}(\mathbf{1}_{[-K,K]}\Phi_{\mu_3}/\Phi_{\mu_1})$$

Then, suppose there exist $b_i, \gamma_i, s_i > 0$ with

- $a \leq |\Phi_{\mu_1}(t)|(1+t^2)^{\gamma_1/2}\exp(b_1|t|^{s_1}) \leq A$, $a, A > 0$
- $\int_{\mathbb{R}} |\Phi_{\mu_2}(t)|^2(1+t^2)^{\gamma_2}\exp(2b_2|t|^{s_2}) < L$ with $L < \infty$,
- $\|d\mu_3 - d(\mu_1 * \mu_2)\|_{L^2} \leq \epsilon$.

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- $\|d\mu_3 - d(\mu_1 * \mu_2)\|_{L^2} \leq \epsilon$.

Then, an appropriate $K = K(\epsilon)$ yields $\|d\widehat{\mu_2} - d\mu_2\|_{L^2}^2$ bounded by a constant $C(L, A)$ times

	$s_1 = 0$	$s_1 > 0$
$s_2 = 0$	$\epsilon^{\frac{2\gamma_2}{2\gamma_1+2\gamma_2+1}}$	$(-\log\epsilon)^{-2s_2/s_1}$
$s_2 > 0$	$(-\log\epsilon)^{\frac{2\gamma_1+1}{b_2}}\epsilon$	complicated, if $s_1 = s_2$: $\epsilon^{\frac{b_2}{b_2+b_1}}(-\log(\epsilon))^{\xi}$

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Optimal bounds !

The classical/free Cauchy deconvolution :

(Butucea, Fan, Lacour and al.)

If $\mu_1 = \mathcal{C}_\lambda$, $\Phi_{\mu_1}(t) = \exp(-\lambda|t|)$.
Hence,

- $a_1 = 1, \gamma_1 = 0$, $b_1 = \lambda$, $s_1 = 1$.
- $\int_{\mathbb{R}} |\Phi_{\mu_2}(t)|^2 (1+t^2)^{\gamma_2} \exp(2b_2|t|^{s_2}) < L$ with $L < \infty$.
- $\|d\mu_3 - d(\mu_1 * \mu_2)\|_{L^2} \leq \epsilon$.

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	$s_1 = 0$	$s_1 > 0$
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$s_2 > 0$	$(-\log \epsilon)^{\frac{2\gamma_1+1}{b_2}} \epsilon$	complicated, if $s_1 = s_2$: $\epsilon^{\frac{b_2}{b_2+b_1}} (-\log(\epsilon))^{\xi}$

The classical classical/free Cauchy deconvolution :

For us : $\mu_1 = \mathcal{C}_{2\sqrt{2}\sigma_1}$, $\Phi_{\mu_1}(t) = \exp(-\lambda|t|)$, $\mu_2 = \mu_A$ and $d\mu_3 = \widehat{C}$.
Hence, if

$$\|\widehat{C} - d(\mathcal{C}_{2\sqrt{2}\sigma} * \mu_A)\|_{L^2} \leq \epsilon$$

an appropriate $K = K(\epsilon)$ yields that $\|d\widehat{\mu_A} - d\mu_A\|_{L^2}^2$ is bounded by

$d\mu_A$ is C^k	$C(k)(-\log\epsilon)^{-2}$
$d\mu_A$ can be analytically extended to $\mathbb{R}\times]-b_2, b_2[$	$\epsilon^{\frac{b_2}{b_2+2\sqrt{2}\sigma}}(-\log(\epsilon))^{\xi}$

There is a version for atomic measures : the condition to ensures well recovery is a minimum separation distance $t(\lambda)$ between atoms of μ_2.

Main goal : to estimate $\|\widehat{C} - d(\mathcal{C}_{2\sqrt{2}\sigma} * \mu_A)\|_{L^2}$.

Convergence result on the subordination method

Recall : $W = UXU^* + A$, U Haar unitary, $A, X \in M_N(\mathbb{C})$ self-adjoint.

Theorem

Suppose that $N \geq K_1$. Then,

$$\mathbb{E}\left(\|\widehat{C} - d(\mathcal{C}_{2\sqrt{2}\sigma_1} * \mu_A)\|_{L^2}^2\right) \leq \frac{K_2}{N^2} + \frac{K_3}{N^3} + \frac{K_4}{N^4},$$

with K_1, K_2, K_3, K_4 depending explicitly on the first six moments of A and X.

Corollary

Suppose that $d_{Levy}(\mu_A, \mu) \leq \frac{1}{N}$, with $d\mu$ analytically extendable to $\mathbb{R} + i]-\tau, \tau[$, $\tau \in]0, \infty]$, then for $N \geq K_1$,

$$\mathbb{P}(\|\widehat{\mu_A} - \mu_A\|_{L^2} > \delta) \leq \frac{C}{\delta^2}\left(\frac{K_2}{N^2} + \frac{K_3}{N^3} + \frac{K_4}{N^4}\right)^{\frac{\tau}{\tau + 2\sqrt{2}\sigma_1}},$$

for some C depending on the regularity of μ.

Convergence result on the subordination method : simulations

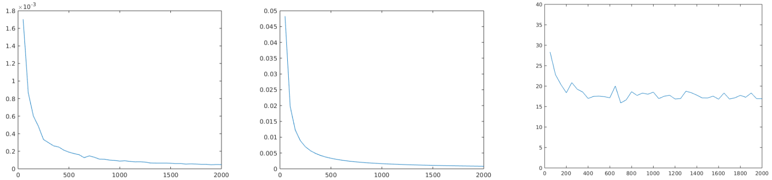

Figure: Simulation of $\|\widehat{C} - d(\mathcal{C}_{2\sqrt{2}\sigma_1} * \mu_A)\|_{L^2}$ in the additive case for N from 50 to 2000 (with a sampling of size 100 for each size), theoretical bound in Theorem, and ratio of the theoretical bound on the simulated error.

Convergence result on the subordination method : simulations

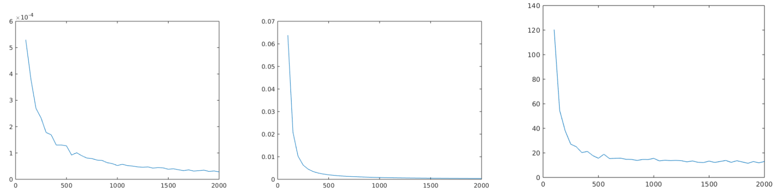

Figure: Simulation of $\|\widehat{C} - d(\mathcal{C}_\kappa * \mu_A)\|_{L^2}$ in the multiplicative case for N from 100 to 2000 (with a sampling of size 100 for each size) , theoretical bound in Theorem, and ratio of the theoretical bound on the simulated error.

Matricial subordination in the additive case

(Pastur and Vasilchuk, Kargin)

$$W = UXU^* + A$$

[For $R \in M_N(\mathbb{C}), R = R^*,\ G_R(z) = (z - R)^{-1},\ G_{\mu_R}(z) = \operatorname{tr}((R - z)^{-1})$]
Set $f_A(z) = -\operatorname{tr}(AG_W(z))$ and $f_X(z) = -\operatorname{tr}(XG_W(z))$. Then, set

$$\omega_A(z) = z + \frac{\mathbb{E}(f_X(z))}{\mathbb{E}(G_{\mu_W}(z))},\ \omega_X(z) = z + \frac{\mathbb{E}(f_A(z))}{\mathbb{E}(G_{\mu_W}(z))}. \qquad (1)$$

The functions ω_A, ω_B are called matricial subordination functions.

$\mu_3 = \mu_A \boxplus \mu_X$	$W = UXU^* + A$
$\tilde\omega_A(z) + \tilde\omega_X(z) = z + \frac{1}{G_{\mu_3}(z)}$	$\omega_A(z) + \omega_X(z) = z + \frac{1}{\mathbb{E}G_{\mu_W}(z)}$
$G_{\mu_3}(z) = G_{\mu_A}(\tilde\omega_A(z)) = G_{\mu_X}(\tilde{\omega_X}(z))$	$\mathbb{E}G_{\mu_W}(z) = G_{\mu_A}(\tilde\omega_A(z)) + R_A(z)$ $= G_{\mu_X}(\tilde{\omega_X}(z)) + R_X(z)$

1. Multiplicative case ? $(XUAU^*X - z)^{-1} \rightsquigarrow X^{-1}(UAU^* - zX^2)^{-1}X^{-1}$
2. Estimates on $R_A(z), R_X(z)$? Estimates on $|G_{\mu_W} - \mathbb{E}G_{\mu_W}|$?

Matricial subordination in the additive case

$$R_A(z) = \frac{\mathbb{E}\left[(G_{\mu_W} - \mathbb{E}G_{\mu_W})(\phi - \mathbb{E}\phi)\right] - \mathbb{E}\left[(f_X - \mathbb{E}f_X)(\psi - \mathbb{E}\psi)\right]}{\mathbb{E}G_{\mu_W}},$$

with $\phi = \mathrm{tr}(G_A(\omega_A(z))UXU^*\tilde{G}_H)$ and $\psi = \mathrm{tr}(G_A(\omega_A(z))G_H)$.

- Using Nevanlinna theory and Weingarten formulas gives an upper bound on $\frac{1}{\mathbb{E}G_{\mu_W}}$ in terms of first moments of A and X,
- Using Poincaré inequality on U_N, matrix Hölder inequalities and Weingarten formulas give an upper bound on $Var(G_{\mu_W})$, $Var(f_X)$, $Var(\phi)$, $Var(\psi)$ in terms of the first moments of X and A. [Reminder of Poincaré inequalities : if $f : U_N \to \mathbb{R}$ has mean zero, then $\int_{U_N} |f|^2 \leq \frac{1}{N} \int_{U_N} \|\nabla f\|^2$]

Finally,

$$|R_A(z)| \leq \frac{K(\Im z)}{N^2},$$

where K depends on the first moments of A and X.

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L^2-estimates

We finally get for $\Im(z) \geq 2\sqrt{2Var(\mu_0)}$

$$|G_{\mu_A}(z) - G_W(\omega_3)| \leq \frac{C_1(\Im z)}{|z|N^2} + \frac{C_2(\Im z)|G_W(\omega_3) - \mathbb{E}G_W(\omega_3)|}{|z|}.$$

Recall

- $\mathbb{E}(|G_W(\omega_3(z)) - \mathbb{E}G_W(\omega_3(z))|^2) \leq \frac{C(\Im z)}{N^2}$,
- $t \mapsto \frac{1}{|t+i\eta|} \in L^2(\mathbb{R})$.

Thus

$$\mathbb{E}\left(\int_{\mathbb{R}} \left| \Im G_W(\omega_3(t + 2\sqrt{2Var(\mu_0)}i)) - \Im G_{\mu_A}(t + 2\sqrt{2Var(\mu_0)}i) \right|^2 dt \right)$$
$$\leq \frac{K_2}{N^2} + \frac{K_3}{N^3} + \frac{K_4}{N^4}.$$

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Thus,

$$\mathbb{E}\left(\|\widehat{C} - d(\mu_A * \mathcal{C}_{2\sqrt{2Var(\mu_0)}}\|_{L^2}^2\right) \leq \frac{K_2}{N^2} + \frac{K_3}{N^3} + \frac{K_4}{N^4}.$$

THANK YOU !