Semi-multiplicative functions and a Hopf Algebra on non-crossing partitions

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Probabilistic Operator Algebra Seminar, UC Berkeley
October 4, 2021

Joint work with Celestino, Ebrahimi-Fard, Nica, Witzman
(Preprint arXiv:2106.16072)

We will discuss

1. Construct a Hopf Algebra $\mathcal{T}$ whose character group $\mathbb{X}(\mathcal{T})$ is naturally isomorphic to $\widetilde{\mathcal{G}}$.

2. Provide a formula for the antipode of $\mathcal{T}$.

3. Observe that the inclusion $\mathcal{G} \subset \widetilde{\mathcal{G}}$ is the dual of a natural surjective homomorphism from $\mathcal{T}$ onto Sym.

The Bialgebra $\mathcal{T}$

Motivation

Recall that, the values of a semi-multiplicative function f are completely determined by set of values

$$\Big\{ f(\pi, 1_n) : n \in \mathbb{N},\ \pi \in NC(n) \setminus \{1_n\} \Big\}.$$

So the convolution of two semi-multiplicative functions $g, f : NC^{(2)} \to \mathbb{C}$ is determined by

$$\begin{aligned} g \star f(\pi, 1_n) &= \sum_{\substack{\sigma \in NC(n) \\ \sigma \geq \pi}} g(\pi, \sigma) f(\sigma, 1_n) \\ &= \sum_{\substack{\sigma \in NC(n) \\ \sigma \geq \pi}} \left(\prod_{V \in \sigma} g(\pi|_V, 1_{|V|}) \right) f(\sigma, 1_n) \end{aligned}$$

This suggests a natural bialgebra structure with indeterminates indexed by non-crossing partitions and the comultiplication dictated by convolution $\star$.

Algebra structure

Let $NC_{\geq 2} := \sqcup_{n=1}^{\infty}\big(NC(n) \setminus \{1_n\}\big)$. Then $\mathcal{T}$ is the commutative algebra of polynomials over $\mathbb{C}$ with indeterminates indexed by non-crossing partitions:

$$\mathcal{T} := \mathbb{C}\Big[\, X_\pi \Big| \pi \in NC_{\geq 2} \,\Big].$$

We make the convention that $X_{1_n} := 1_{\mathcal{T}}$, $\forall\, n \geq 1$.

Universality property: If $\mathcal{A}$ is a unital commutative algebra over $\mathbb{C}$ and we are given elements $\{a_\pi | \pi \in NC\}$ with $a_{1_n} = 1_{\mathcal{A}}$ for all $n \geq 1$, then there exists a unital algebra homomorphism $\Phi : \mathcal{T} \to \mathcal{A}$, uniquely determined, such that $\Phi(X_\pi) = a_\pi$ for all $\pi \in NC$.

In particular, characters $\chi \in \mathbb{X}(\mathcal{T})$ (uni. alg. hom. $\chi : \mathcal{T} \to \mathbb{C}$) are in bijection with the semi-multiplicative functions $g \in \widetilde{\mathcal{G}}$. We simply take $\chi(X_\pi) = g(\pi, 1_n)$.

Bialgebra structure

- **Counit** $\epsilon : \mathcal{T} \to \mathbb{C}$. The uni. alg. hom. determined by

$$\epsilon(X_\pi) = 0, \qquad \forall \pi \in NC_{\geq 2},$$

- **Comultiplication** $\Delta : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$. The uni. alg. hom.

$$\Delta(X_\pi) = \sum_{\substack{\sigma \in NC(n), \\ \sigma \geq \pi}} \Big(\prod_{W \in \sigma} X_{\pi_W} \Big) \otimes X_\sigma, \qquad \forall \pi \in NC.$$

Δ is coassociative and compatible with counit, so $\mathcal{T}$ is Bialgebra.

Why take this approach?

–$\mathcal{T}$ has an antipode that can be used to systematically compute inverses.

–Analogy to how the Hopf algebra Sym is used in the study of the multiplication of free random variables [Mastnak, Nica 2010].

The antipode of $\mathcal{T}$

$\mathcal{T}$ is a Hopf Algebra

We let the indeterminate X_π have degree $|\pi| - 1$ and naturally extend to monomials so that $\deg(X_{\pi_1} \cdots X_{\pi_k}) = |\pi_1| + \cdots + |\pi_k| - k$.

This makes $\mathcal{T} = \bigoplus_{n=0}^{\infty} \mathcal{T}_n$ a **graded** bialgebra:

$$\mathcal{T}_n \cdot \mathcal{T}_m \subset \mathcal{T}_{m+n} \qquad \text{and} \qquad \Delta(\mathcal{T}_n) \subset \sum_{m=0}^{n} \mathcal{T}_m \otimes \mathcal{T}_{n-m}$$

Notice that $\mathcal{T}_0 := \{\lambda \cdot 1_{\mathcal{T}} \mid \lambda \in \mathbb{C}\}$, so $\mathcal{T}$ is **connected**.

Since $\mathcal{T}$ is a graded connected bialgebra, then $\mathcal{T}$ is a **Hopf algebra**. Then it has **antipode** $S : \mathcal{T} \to \mathcal{T}$, uni. alg. hom. such that

$$mult \circ (Id \otimes S) \circ \Delta = \epsilon 1_{\mathcal{T}} = mult \circ (S \otimes Id) \circ \Delta$$

S is the inverse under convolution of the identity map in $L(\mathcal{T})$.
Most important fact: $\chi^{-1} = \chi \circ S$

Bogoliubov formulas

We can recursively compute the antipode with the formulas

$$S(X_\pi) = -X_\pi - \sum_{\substack{\sigma \in NC(n), \\ \pi < \sigma < 1_n}} \left(\prod_{W \in \sigma} X_{\pi_W} \right) S(X_\sigma),$$

$$S(X_\pi) = -X_\pi - \sum_{\substack{\sigma \in NC(n), \\ \pi < \sigma < 1_n}} \left(\prod_{W \in \sigma} S(X_{\pi_W}) \right) X_\sigma,$$

Given $\pi, \sigma \in NC(n)$ with $\pi < \sigma$. A *chain* from π to σ is a tuple

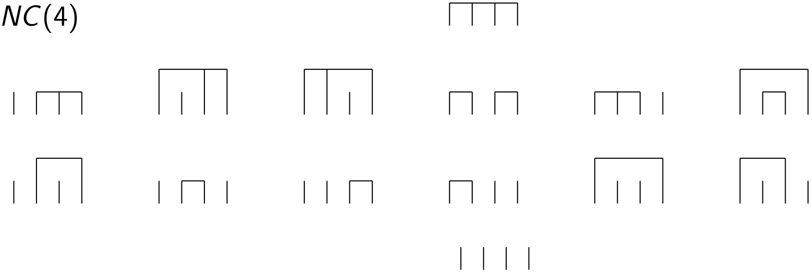
$$c = (\pi_0, \pi_1, \ldots, \pi_k), \text{ where } \pi = \pi_0 < \pi_1 < \cdots < \pi_k = \sigma.$$

We have also an explicit formula [Schmitt 1987] in terms of chains.

$$S(X_\pi) = \sum_{\substack{c = (\pi_0, \pi_1, \ldots, \pi_k), \\ \text{chain from } \pi \text{ to } 1_n}} (-1)^k \prod_{i=0}^{k-1} \prod_{V \in \pi_{i+1}} X_{\pi_i|_V}.$$

There are cancellations

$NC(4)$



Term of $c = ($  $)$:

If $c = ($  $)$, then we do

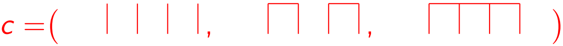
Same for $c = ($  $)$.

Efficient chains and cancellation free formula

For $c = (\pi_0, \pi_1, \ldots, \pi_k)$, denote $\mathrm{Blocks}^+(c) := (\pi_1 \cup \pi_2 \cup \cdots \cup \pi_k) \setminus \pi_0$.

A chain c is **efficient** when for every block $V \in \mathrm{Blocks}^+(c)$ there exists a *unique* $j \in \{1, \ldots, k\}$ such that $V \in \pi_j$.


not efficient.


efficient.

Theorem (Celestino, Ebrahimi-Fard, Nica, Perales, Witzman)

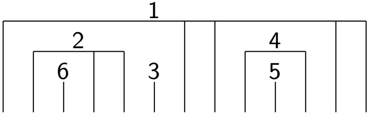
For every $\pi \in NC_{\geq 2}$ we have a cancellation-free formula

$$S(X_\pi) = \sum_{\substack{\text{efficient chain} \\ c=(\pi_0,\pi_1,\ldots,\pi_k) \\ \text{from } \pi \text{ to } 1_n}} (-1)^{|\mathrm{Blocks}^+(c)|} \prod_{i=0}^{k-1} \prod_{V \in \pi_{i+1}} X_{\pi_i|_V}.$$

Application: Boolean-to-monotone cumulant formula

Idea: use S to invert the monotone-to-Boolean cumulant formula.

$$\beta_n = \sum_{(\pi,\lambda)\in\mathcal{M}_{irr}(n)} \frac{1}{|\pi|!}\rho_\pi$$



Let $mon(\pi)$ be the number of monotone labelings of π, then

$$\beta_n = \sum_{\pi\in NC_{irr}(n)} \frac{mon(\pi)}{|\pi|!}\rho_\pi,$$

so we have

$$\chi_{\mathrm{mc-bc}}(X_\pi) = g_{\mathrm{mc-bc}}(\pi, 1_n) = \begin{cases} \frac{mon(\pi)}{|\pi|!} & \text{if } \pi \text{ is irreducible,} \\ 0 & \text{otherwise} \end{cases}$$

Application: Boolean-to-monotone cumulant formula

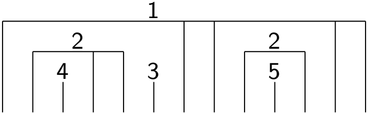
For every $\pi \in NC$, we have

$$\chi_{\mathrm{bc-mc}}(X_\pi) = \sum_{\substack{\text{efficient chain} \\ c=(\pi_0,\pi_1,\ldots,\pi_k) \\ \text{from } \pi \text{ to } 1_n}} (-1)^{|\mathrm{Blocks}^+(c)|} \prod_{i=1}^{k} \prod_{V\in\pi_{i+1}} \chi_{\mathrm{mc-bc}}(X_{\pi_i|_V})$$

$$= \sum_{\substack{\text{efficient chain} \\ c=(\pi_0 \ll \pi_1 \ll \ldots \ll \pi_k) \\ \text{from } \pi \text{ to } 1_n}} (-1)^{|\mathrm{Blocks}^+(c)|} \prod_{i=1}^{k} \prod_{V\in\pi_{i+1}} \frac{mon(\pi_i|_V)}{|\pi_i|_V|!}.$$

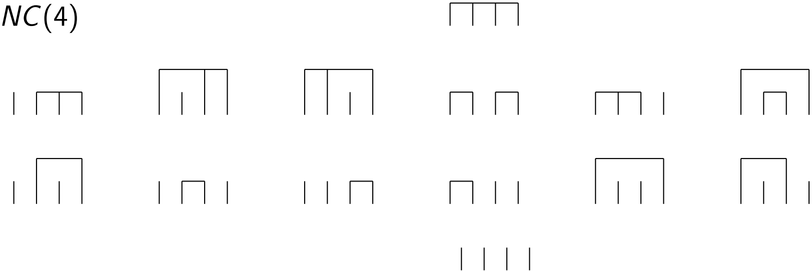
For comparison, in [Celestino, Ebrahimi-Fard, Patras, Perales, 2021]

$$\chi_{\mathrm{bc-mc}}(X_\pi) = \sum_{k=1}^{n} \frac{(-1)^k}{k} \omega_k(\pi),$$



where $\omega_k(\pi) :=$ number of strictly decreasing k-colourings of π.

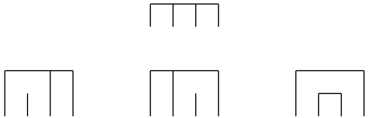
$NC(4)$



$\chi_{\mathrm{bc-mc}}(X_\pi) = 0$ if π is reducible

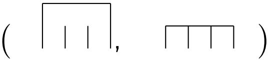
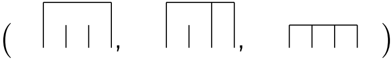
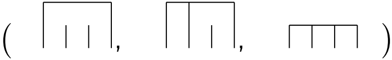
Low order example

$NC_{irr}(4)$



$\chi_{\mathrm{bc-mc}}(X_\pi)$

$\pi =$ 

Chain	Monomial (M)	$\chi_{\mathrm{mc-bc}}(M)$
		$-\frac{2}{3!}$
		$+\frac{1}{2!}\cdot\frac{1}{2!}$
		$+\frac{1}{2!}\cdot\frac{1}{2!}$

$$\chi_{\mathrm{bc-mc}}(X_\pi) = -1/3 + 1/4 + 1/4 = 1/6$$

Surjective homomorphism from $\mathcal{T}$ to Sym

Hopf algebra Sym

Sym $\simeq \mathbb{C}[E_1, E_2, \ldots, E_n, \ldots]$ is the algebra of symmetric polynomials in countable family of (commuting) indeterminates $(t_i)_{i=1}^{\infty}$. It is generated by the *elementary symmetric functions*

$$E_n := \sum_{1 \leq i_1 < \cdots < i_n} t_{i_1} t_{i_2} \cdots t_{i_n},$$

Sym is a Hopf algebra:

- **Counit.** $\epsilon : \mathrm{Sym} \to \mathbb{C}$, with $\epsilon(E_n) = 0$ for all $n \geq 1$.
- **Comultiplication.** $\Delta : \mathrm{Sym} \to \mathrm{Sym} \otimes \mathrm{Sym}$,

$$\Delta(E_n) = \sum_{i=0}^{n} E_i \otimes E_{n-i}, \qquad \forall\, n \geq 0 \;\; (\text{with } E_0 := 1_{\mathrm{Sym}}).$$

- **Grading.** $\deg(E_n) = n$

The generators Y_n for Sym

For the connection between $\boxtimes$ and Sym, we need another set of generators for Sym, which we call $(Y_n)_{n=1}^{\infty}$, defined by $Y_1 := 1_{\mathrm{Sym}}$ and

$$Y_n := \sum_{\pi \in NC(n-1)} E_\pi$$

The Y_n's are called *parking function symmetric functions*. E.g.

$$Y_1 = 1, \quad Y_2 = E_1, \quad Y_3 = E_2 + E_1^2, \quad Y_4 = E_3 + 3E_1E_2 + E_1^3, \dots$$

$$\begin{array}{rlll} \textbf{Grading} & \deg(Y_n) & = n-1, & \forall n \geq 0, \\ \textbf{Counit} & \epsilon(Y_n) & = 0, & \forall n \geq 1, \\ \textbf{Comultiplication} & \Delta(Y_n) & = \sum_{\pi \in NC(n)} Y_\pi \otimes Y_{Kr(\pi)}, & \forall n \geq 0, \end{array}$$

$$\begin{aligned} \Delta(Y_1) &= 1 \otimes 1 = Y_1 \otimes Y_1, \\ \Delta(Y_2) &= 1 \otimes E_1 + E_1 \otimes 1 = Y_1^2 \otimes Y_2 + Y_2 \otimes Y_1^2, \\ \Delta(Y_3) &= \Delta(E_2) + (\Delta(E_1))^2 = Y_1^3 \otimes Y_3 + 3Y_1Y_2 \otimes Y_1Y_2 + Y_3 \otimes Y_1^3. \end{aligned}$$

Homomorphism $\Psi : \mathcal{T} \to \text{Sym}$

Theorem

The uni. alg. hom. $\Psi : \mathcal{T} \to Sym$ defined by

$$\Psi(X_\pi) = Y_{\mathrm{Kr}(\pi)} = \prod_{W \in \mathrm{Kr}(\pi)} Y_{|W|}, \qquad \forall \pi \in NC,$$

is a surjective homomorphism of graded connected bialgebras.

Corollary

There is an injective group homomorphism $\Psi^* : \mathbb{X}(Sym) \to \mathbb{X}(\mathcal{T})$,

$$\Psi^*(\chi) := \chi \circ \Psi, \qquad \chi \in \mathbb{X}(Sym).$$

So, when we identify $\mathbb{X}(Sym) \sim \mathcal{G}$ and $\mathbb{X}(\mathcal{T}) \sim \widetilde{\mathcal{G}}$ we retrieve the inclusion of $\mathcal{G}$ into $\widetilde{\mathcal{G}}$.

Future stuff to look at

- Hopf algebra constructions for some relevant subgroups of $\widetilde{\mathcal{G}}$. We already have this for $\widetilde{\mathcal{G}}_{\mathrm{c-c}}$.

- We know another Hopf algebra (of trees) that can be obtained from $\mathcal{T}$ (using the nesting tree structure of a partition).

- The T-free cumulants [Jekel, Liu 2019] can be studied with this framework. Is there a deeper connection.

- Study more general transition formulas (like infinitesimal, conditional).

Thank you!