Positive Maps and Entanglement in Real Hilbert Spaces

Vern Paulsen
joint with G. Chiribella, K. Davidson, and M. Rahaman

October 10, 2022

Motivations

What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers. Ψ is surely fundamentally a real function.
Letter from Schrodinger to Lorentz. June 6th, 1926.

Even though quantum theory uses complex Hilbert spaces and they play a key "tidying" role in the theory, it is only fairly recently that physicists have started to ask if the quantum world is inherently complex.
The work of M. Navascues et al has shown that many parts of quantum theory remain unchanged if the underlying Hilbert spaces are all assumed to be real.
This includes all the various types of *quantum correlation sets*, which were central to the recent proof that the Connes Embedding Problem has a negative answer.

In *Quantum theory based on real numbers can be experimentally falsified*(Navascues et al), Dec. 27, 2021, a Bell-like experiment based on a network scenario is proposed that numerically separates complex from real quantum theory.

Trials of this experiment have now shown this separation.

In brief, it has now been shown that the real-world is not!

Other labs are trying to replicate this experiment and theoreticians are looking for other, possibly better, separations.

Our work looks at the similarities/differences between the real and complex case for various concepts like, entanglement and separability, positive maps, p-positive maps and the various characterizations of entanglement breaking maps.

Notation

We let $\mathbb{K}$ denote the real $\mathbb{R}$ or complex $\mathbb{C}$ numbers, $\mathbb{K}^n$ the vector space of n-tuples, which is a Hilbert space in the usual inner product.
$M_{n,k}(\mathbb{K})$ denotes the $n \times k$ matrices with entries from $\mathbb{K}$, which we identify with $L(\mathbb{K}^k, \mathbb{K}^n)$.
P^t denotes the transpose of a matrix and P^* the conjugate transpose.

$$Herm_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) : A = A^*\},$$
$$Sym_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) : A = A^t\},$$
$$Asym_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) : A^t = -A\},$$
$$PSD_n(\mathbb{K}) = \{X \in M_n(\mathbb{K}) : X = Y^*Y\}$$

$$M_n(\mathbb{C}) = Herm_n(\mathbb{C}) + iHerm_n(\mathbb{C})$$

and

$$Herm_n(\mathbb{C}) = PSD_n(\mathbb{C}) - PSD_n(\mathbb{C}),$$

while

$$M_n(\mathbb{R}) = Sym_n(\mathbb{R}) \oplus Asym_n(\mathbb{R})$$

and

$$Sym_n(\mathbb{R}) = PSD_n(\mathbb{R}) - PSD_n(\mathbb{R}).$$

Bipartite Systems and Block Matrices

We let $E_{i,j}$ denote the matrix that is 1 in the (i,j)-entry and 0 elsewhere so that $\{E_{i,j} : 1 \le i \le k, 1 \le j \le k\}$ is a basis for $M_k(\mathbb{K})$.
We let $M_k(M_n(\mathbb{K})) = \{(A_{i,j}) : A_{i,j} \in M_n(\mathbb{K})\}$ denote the set of $k \times k$ matrices whose entries are $n \times n$ matrices and identify

$$M_k(\mathbb{K}) \otimes M_n(\mathbb{K}) \simeq M_k(M_n(\mathbb{K})) \simeq M_{kn}(\mathbb{K})$$

where the first map is defined via $\sum_{i,j} E_{i,j} \otimes A_{i,j} \simeq (A_{i,j})$ and the second map is by erasing the extra parentheses. We set

$$Psd(\mathbb{K}^k, \mathbb{K}^n) = \{(A_{i,j}) \in M_k(M_n(\mathbb{K})) : (A_{i,j}) \in PSD_{kn}(\mathbb{K})\}.$$

Separable and Entangled

We set

$$SEP(\mathbb{K}^k, \mathbb{K}^n) := \{\sum_l P_l \otimes Q_l : P_l \in PSD_k(\mathbb{K}), Q_l \in PSD_n(\mathbb{K})\}.$$

Note that with our identifications,

$$SEP(\mathbb{K}^k, \mathbb{K}^n) \subseteq PSD(\mathbb{K}^k, \mathbb{K}^n).$$

Any matrix in $PSD(\mathbb{K}^k, \mathbb{K}^n) \backslash SEP(\mathbb{K}^k, \mathbb{K}^n)$ is called $\mathbb{K}$-*entangled*. If a real matrix is separable as a complex matrix, is it necessarily also separable as a real matrix? That is, is

$$SEP(\mathbb{R}^k, \mathbb{R}^n) \stackrel{?}{=} SEP(\mathbb{C}^k, \mathbb{C}^n) \cap M_k(\mathbb{R}) \otimes M_n(\mathbb{R}) := CSEP(\mathbb{R}^k, \mathbb{R}^n).$$

NO!

$SEP(\mathbb{R}^k, \mathbb{R}^n) \subsetneq CSEP(\mathbb{R}^k, \mathbb{R}^n)$

Let $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ and set

$$X = I_2 \otimes I_2 + A \otimes A = \left(\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right) \in \mathrm{PSD}(\mathbb{R}^2, \mathbb{R}^2).$$

Since $I_2 \pm A \geq 0$ and

$$2X = (I_2 + A) \otimes (I_2 + A) + (I_2 - A) \otimes (I_2 - A),$$

this matrix is $\mathbb{C}$-separable.
But note that if $X = \sum_l P_l \otimes Q_l$ is $\mathbb{R}$-separable, then $id \otimes T(X) = \sum_l P_l \otimes T(Q_l) = X$. Since $id \otimes T(X) \neq X$, X is not $\mathbb{R}$-separable.

Dual Cones and Entanglement Witnesses

Another identification that we shall use is the *Choi-Jamiolkowska isomorphism*. This identifies $L(M_d, M_r) \simeq M_d \otimes M_r$ via

$$\Phi \to C_\Phi := \sum_{i,j=1}^{d} E_{i,j} \otimes \Phi(E_{i,j}).$$

A map $\Phi : M_d(\mathbb{K}) \to M_r(\mathbb{K})$ is called **positive** provided that $\Phi(PSD_d(\mathbb{K})) \subseteq PSD_r(\mathbb{K})$ and $\Phi(X^*) = \Phi(X)^*$.

Stormer's Theorem: Let $\Phi : M_d(\mathbb{C}) \to M_r(\mathbb{C})$ be linear. Then Φ is positive if and only if

$$Tr(C_\Phi X) \geq 0, \forall X \in SEP(\mathbb{C}^d, \mathbb{C}^r).$$

In particular, if X is $\mathbb{C}$-entangled, then there is a positive map such that $Tr(C_\Phi X) < 0$.
The proof and result in the real case is identical:

Theorem: Let $\Phi : M_d(\mathbb{R}) \to M_r(\mathbb{R})$ be linear. Then Φ is positive if and only if

$$Tr(C_\Phi X) \geq 0, \forall X \in SEP(\mathbb{R}^d, \mathbb{R}^r).$$

In particular, if X is $\mathbb{R}$-entangled, then there is a positive map such that $Tr(C_\Phi X) < 0$.

Note that every $X \in M_n(\mathbb{C})$ can be written uniquely as $X = A + iB$ with $A, B \in M_n(\mathbb{R})$. Given a real linear map $\Phi : M_d(\mathbb{R}) \to M_r(\mathbb{R})$ its *complexification* is the complex linear map $\tilde{\Phi} : M_d(\mathbb{C}) \to M_r(\mathbb{C})$ given by

$$\tilde{\Phi}(A + iB) = \Phi(A) + i\Phi(B).$$

Theorem: Let $\Phi : M_d(\mathbb{R}) \to M_r(\mathbb{R})$ be linear. Then $\tilde{\Phi}$ is positive if and only if

$$Tr(C_\Phi X) \geq 0, \forall X \in CSEP(\mathbb{R}^d, \mathbb{R}^r).$$

In particular, if $X \in CSEP(\mathbb{R}^d, \mathbb{R}^r) \backslash SEP(\mathbb{R}^d, \mathbb{R}^r)$, then there is a positive map Φ such that $\tilde{\Phi}$ is not positive and $Tr(C_\Phi X) < 0$.

Since $SEP(\mathbb{R}^2, \mathbb{R}^2) \subsetneq CSEP(\mathbb{R}^2, \mathbb{R}^2)$ the last result implies that there must exist positive maps, whose complexification is not positive.

Here is a family of such maps. For $0 < s < 1$ define $\Phi_s : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $\Phi_s(X) = X - sX^t$. If $X \in PSD_2(\mathbb{R})$, then $X = X^t$ and so $\Phi_s(X) = (1-s)X \geq 0$. Thus, Φ_s is a positive map. But its complexification is seen to be the map $\tilde{\Phi_s} : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ given by the same formula.

Note $P = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \in PSD_2(\mathbb{C})$ and

$$\Phi_s(P) = \begin{pmatrix} 1-s & (1+s)i \\ -(1+s)i & 1-s \end{pmatrix} \notin PSD_2(\mathbb{C}).$$

State and Channels

When we say that $\mathbb{K}^n$ is the state space of a quantum system it means that the *pure states* of the system are represented by unit vectors $\psi \in \mathbb{K}^n$, or more precisely that each pure state is represented by the rank one projection onto the span of ψ,

$$P_\psi = |\psi >< \psi| = \psi\psi^*.$$

A *state ensemble* or *mixed state* is a set of unit vectors along with probability densities, $\{\psi_i, p_i\}$ and these are represented by *density matrices*,

$$\sum_i p_i P_{\psi_i},$$

which is a PSD matrix of trace one. Conversely, every PSD matrix of trace one is the density matrix of some state ensemble.

A *quantum channel* is a linear map $\Phi : M_d(\mathbb{K}) \to M_r(\mathbb{K})$ such that for all n,

$$id_n \otimes \Phi : M_n(\mathbb{K}) \otimes M_d(\mathbb{K}) \to M_n(\mathbb{K}) \otimes M_r(\mathbb{K}),$$

maps density matrices to density matrices. It is not hard to see that Φ is a quantum channel if and only if Φ is completely positive(CP) and trace preserving(TP) where TP means $Tr(\Phi(X)) = Tr(X), \forall X$.
Thus, quantum channel and CPTP are synonymous.

Choi's Theorem: Let $\Phi : M_d(\mathbb{K}) \to M_r(\mathbb{K})$ be linear. Then Φ is CP if and only if $C_\Phi \in PSD(\mathbb{K}^d, \mathbb{K}^r)$.
Moreover, if Φ is CPTP, then $\frac{1}{d} C_\Phi$ is a density matrix.
So often channels are identified with bipartite states.
Since separable and entangled are important, it is interesting to unravel what these properties mean in terms of channels.

Entanglement Breaking Maps

Definition: A CP map $\Phi : M_d(\mathbb{K}) \to M_r(\mathbb{K})$ is called $\mathbb{K}$-**entanglement breaking** if for every n and every $P = (P_{i,j}) \in PSD(\mathbb{K}^n, \mathbb{K}^d)$ we have that

$$id_n \otimes \Phi(P) = (\Phi(P_{i,j})) \in SEP(\mathbb{K}^n, \mathbb{K}^r).$$

Horodecki-Shor-Ruskai: Let $\Phi : M_d(\mathbb{C}) \to M_r(\mathbb{C})$ be a CP map. Then the following are equivalent:

1. Φ is $\mathbb{C}$-entanglement breaking,
2. $C_\Phi \in SEP(\mathbb{C}^d, \mathbb{C}^r)$,
3. there exist rank one matrices $A_i \in M_{r,d}(\mathbb{C})$ such that

$$\Phi(X) = \sum_i A_i X A_i^*,$$

4. (Johnston-Kribs-P-Pereira) there exists k and completely positive maps $\Delta : M_d(\mathbb{C}) \to \ell_k^\infty(\mathbb{C})$ and $\Psi : \ell_k^\infty(\mathbb{C}) \to M_r(\mathbb{C})$ such that $\Phi = \Psi \circ \Delta$,
5. Same statement with $\ell_k^\infty(\mathbb{C})$ replaced by an arbitrary finite dimensional abelian C*-algebra.

Theorem: Let $\Phi : M_d(\mathbb{R}) \to M_r(\mathbb{R})$ be a CP map. Then the following are equivalent:

1. Φ is $\mathbb{R}$-entanglement breaking,
2. $C_\Phi \in SEP(\mathbb{R}^d, \mathbb{R}^r)$,
3. there exist rank one matrices $A_i \in M_{r,d}(\mathbb{R})$ such that
$$\Phi(X) = \sum_i A_i X A_i^t,$$
4. there exists k and completely positive maps $\Delta : M_d(\mathbb{R}) \to \ell_k^\infty(\mathbb{R})$ and $\Psi : \ell_k^\infty(\mathbb{R}) \to M_r(\mathbb{R})$ such that $\Phi = \Psi \circ \Delta$.

Theorem: Let $\Phi : M_d(\mathbb{R}) \to M_r(\mathbb{R})$ be a CP map. Then the following are equivalent:

1. for every n and every $P = (P_{i,j}) \in PSD(\mathbb{R}^n, \mathbb{R}^d)$ we have that
$$id_n \otimes \Phi(P) = (\Phi(P_{i,j})) \in CSEP(\mathbb{R}^n, \mathbb{R}^r).$$
2. $\tilde{\Phi}$ is $\mathbb{C}$-entanglement breaking,
3. $C_\Phi \in CSEP(\mathbb{R}^d, \mathbb{R}^r)$,
4. there exist rank one matrices $A_i \in M_{r,d}(\mathbb{C})$ such that
$$\Phi(X) = \sum_i A_i X A_i^*,$$
5. there exists a finite dimensional abelian real C*-algebra $\mathcal{C}$ and CP maps $\Delta : M_d(\mathbb{R}) \to \mathcal{C}$, $\Psi : \mathcal{C} \to M_r(\mathbb{R})$ such that $\Phi = \Psi \circ \Delta$.

Here by "real C*-algebra", we mean a real subalgebra of an ordinary C*-algebra.

The PPT^2 Conjecture

A map $\Phi : M_d(\mathbb{K}) \to M_r(\mathbb{K})$ is called **PPT**(positive partial transpose) if Φ and $T \circ \Phi$ are both CP, where T denotes the transpose map.

PPT^2 **Conjecture:** If $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is PPT, then $\Phi \circ \Phi$ is EB.

Arises from the study of *quantum repeaters*. Basically if conjecture is true, then PPT maps not useful in quantum repeaters.

Here are a few things that we know:

- It is true for $d = 2, 3$ and for many families of maps.
- It is equivalent to the conjecture that if $\Phi_i : M_{d_i}(\mathbb{C}) \to M_{d_{i+1}}(\mathbb{C})$ are PPT, then $\Phi_2 \circ \Phi_1$ is EB.
- (Kennedy-Manor-P) If Φ is PPT, unital and idempotent, then its range is an abelian C*-algebra in the Choi-Effros product, and hence Φ is EB.
- (Kennedy-Manor-P) If Φ is PPT and unital or trace-preserving, then $\lim_n d(\Phi^n, EB) = 0$.
- (Jaques-P-Rahaman) If Φ is PPT unital and trace-preserving, then there exists k such that Φ^k is EB.

Define $\Phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ with

$$C_\Phi = \frac{1}{2}\begin{bmatrix} I_2 & \gamma \\ -\gamma & I_2 \end{bmatrix} \quad \text{where} \quad \gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then Φ is PPT, unital, trace preserving and idempotent, $\Phi^2 = \Phi$.
Moreover, $range\Phi = span\{I_2, \gamma\}$ is a real abelian C*-algebra with the usual product.
But $\Phi = \Phi \circ \Phi$ is not $\mathbb{R}$-entanglement breaking.
However, $C_\Phi \in CSEP(\mathbb{R}^2, \mathbb{R}^2)$. Therefore $\tilde{\Phi}$ is $\mathbb{C}$-entanglement breaking.

The ITP^2 Conjecture

Seeking a reasonable real version.
A map $\Phi : M_d(\mathbb{R}) \to M_r(\mathbb{R})$ is **ITP** if it is CP and $T \circ \Phi = \Phi$.

Conjecture: If $\Phi : M_d(\mathbb{R}) \to M_d(\mathbb{R})$ is ITP, then $\Phi \circ \Phi$ is $\mathbb{R}$-EB.
Things we know:

- If PPT^2 true, then IPT^2 true.
- Do not know if they are equivalent.
- If Φ ITP, unital and idempotent, then the range of Φ in the Choi-Effros product is isomorphic to $\ell^\infty_k(\mathbb{R})$ for some k.
- If Φ ITP and unital or trace-preserving, then $\lim_n d(\Phi^n, \mathbb{R} - EB) = 0$.
- If Φ ITP, unital and trace-preserving, then there exists k such that Φ^k is $\mathbb{R}$-EB.

Further Results

There is also a definition of *p-separable*, $SEP_p(\mathbb{K}^d, \mathbb{K}^r)$ is the cone generated by rank one operators vv^* where $v \in \mathbb{K}^d \otimes \mathbb{K}^r$ has Schimdt rank at most p.
We prove that $SEP_p(\mathbb{K}^d, \mathbb{K}^r)$ is dual to the cone of C_Φ such that $\Phi : M_d(\mathbb{K}) \to M_r(\mathbb{K})$ is p-positive.

Similar to earlier results, the set of maps such that C_Φ is p-separable are exactly the *p-entanglement breaking maps* and these are the maps such that Φ factors through $\ell_k^\infty(\mathbb{K}) \otimes M_p(\mathbb{K})$.

Thanks!