Asymptotic expansions in Random Matrix Theory and application: the case of Haar unitary matrices

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Based on the following work:

Asymptotic expansion of smooth functions in deterministic and iid Haar unitary matrices, and application to tensor products of matrices

Original motivation

Problem

Given a family $X^N = (X_1^N, \ldots, X_d^N)$ of self-adjoint random matrices, P a noncommutative polynomial, how does the operator norm of $P(X^N)$ behaves asymptotically? I.e. can we compute $\lim_{N\to\infty} \left\|P(X^N)\right\|$?

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A necessary assumption

There exists a family $x = (x_1, \dots, x_d)$ *of self-adjoint elements of a* $\mathcal{C}^*$*-algebra* $\mathcal{A}$ *endowed with a faithful trace* τ *such that almost surely, the family* X^N *converges in distribution towards* x*. That is for any noncommutative polynomial* Q*,*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\left(Q(X^N) \right) = \tau(Q(x)).$$

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Problem

Given a family $X^N = (X_1^N, \dots, X_d^N)$ *of random matrices,* P *a noncommutative polynomial, can we prove that almost surely:*

$$\lim_{N\to\infty} \left\| P(X^N) \right\| = \|P(x)\|?$$

A lower bound on the limit

We know that for any $k \in \mathbb{N}$,

$$\left\| P\left(X^N\right) \right\| \geq \left(\frac{1}{N} \operatorname{Tr}\left(\left(P\left(X^N\right)^* P\left(X^N\right) \right)^k \right) \right)^{1/2k}.$$

Consequently, thanks to the convergence in distribution,

$$\liminf_{N\to\infty} \left\| P(X^N) \right\| \geq \tau\left(\left(P(x)^* P(x) \right)^k \right)^{1/2k}.$$

And since it is known that $\lim_{k\to\infty} \tau\left((P(x)^* P(x))^k \right)^{1/2k} = \|P(x)\|$,

$$\liminf_{N\to\infty} \left\| P(X^N) \right\| \geq \|P(x)\|.$$

An upper bound on the limit

One has for any $k \in \mathbb{N}$ that:

$$\left\| P(X^N) \right\|^{2k} = \left\| \left(P(X^N)^* P(X^N) \right)^k \right\| \leq \mathrm{Tr}\left(\left(P(X^N)^* P(X^N) \right)^k \right).$$

Thus heuristically,

$$\begin{aligned} \left\| P(X^N) \right\| &\leq N^{1/2k} \ \left(\tau\left((P(x)^* P(x))^k \right) \right)^{1/2k} \\ &\leq N^{1/2k} \left\| P(x) \right\|. \end{aligned}$$

Thus one would like to take $k \gg \ln(N)$. However doing so make it way more complicated to control the error term.

A few notions of Free probability

Definition

Let $A = (a_1, \dots, a_k)$ be a k-tuple of elements of a $\mathcal{C}^*$-algebra. The **joint $*$-distribution** of the family A is the linear form

$$\mu_A : P \mapsto \tau\big[P(A, A^*)\big]$$

on the set of polynomials in $2k$ noncommutative variables.

Definition

By **convergence in distribution**, for a sequence of families of variables $(A_N)_{N\geq 1} = (a_1^N, \dots, a_k^N)_{N\geq 1}$ in $\mathcal{C}^*$-algebras $\big(\mathcal{A}_N, ^*, \tau_N, \|.\|\big)$, we mean the pointwise convergence of the map

$$\mu_{A_N} : P \mapsto \tau_N\big[P(A_N, A_N^*)\big].$$

Definition

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Theorem (D. Voiculescu, 1991)

Let $U^N = (U_1^N, \dots, U_d^N)$ be independent Haar unitary matrices, $u = (u_1, \dots, u_d)$ a d-tuple of free Haar unitaries. Then almost surely U^N converges in distribution towards u. That is almost surely for any noncommutative polynomial P,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\Big(P(U^N, (U^N)^*)\Big) = \tau\Big(P(u, u^*)\Big) .$$

Let $(X_t)_{t\geq 0}$ be a Markov process associated with infinitesimal generator $\mathcal{L}$, then it is known that:

- For f in the domain of $\mathcal{L}$, $\frac{d}{dt}\mathbb{E}[f(X_t)] = \mathbb{E}[\mathcal{L}f(X_t)]$.
- If Z is the invariant law of this Markov process, then X_t converges in law towards Z.

Markov Processes in classical stochastic calculus

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Thus,

$$\mathbb{E}[f(Z)] - \mathbb{E}[f(X_0)] = \lim_{t\to\infty} \mathbb{E}[f(X_t)] - \mathbb{E}[f(X_0)] = \int_0^\infty \mathbb{E}[\mathcal{L}f(X_t)]\, dt.$$

In particular if $X_0 \sim Z$, then $\mathbb{E}[f(X_t)]$ is constant and thus for any t, $\mathbb{E}[\mathcal{L}f(X_t)] = 0$.

Let $(x_t)_{t\geq 0}$ be a free Markov process associated with infinitesimal generator $\mathcal{L}$, then it is known that:

- For any polynomial P, $\frac{d}{dt}\tau[P(x_t)] = \tau[\mathcal{L}P(x_t)]$.
- If z is the invariant distribution of this free Markov process, then x_t converges in distribution towards z.

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Thus,

$$\tau[P(x)] - \tau[P(x_0)] = \lim_{t\to\infty} \tau[P(x_t)] - \tau[P(x_0)] = \int_0^{\infty} \tau[\mathcal{L}P(x_t)]\, dt.$$

In particular if $x_0 \sim z$, then $\tau[P(x_t)]$ is constant and thus for any t, $\tau[\mathcal{L}P(x_t)] = 0$.

- Let us assume that there exists a free Markov process whose invariant distribution is the one of u. We set
  - $(u_t^N)_{t\geq 0}$ such a Markov process started in U^N,
  - $(u_t)_{t\geq 0}$ such a Markov process started in u.

  Then after showing that this is well-defined, for any polynomial Q,

$$\tau[Q(u)] - \mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left[Q(U^N)\right]\right] = \int_0^\infty \mathbb{E}\left[\tau_N\left[\mathcal{L}Q(u_t^N)\right]\right]\ dt.$$

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- Heuristically since we know that U^N converges in distribution towards u, then

$$\lim_{N\to\infty} \mathbb{E}\left[\tau_N\left[\mathcal{L}Q(u_t^N)\right]\right] = \tau\left[\mathcal{L}Q(u_t)\right] = 0.$$

Theorem (P., 2023)

Let the following objects be given,

- $U^N = (U_1^N, \dots, U_d^N)$ *independent Haar unitary matrices in* $\mathbb{M}_N(\mathbb{C})$,
- P *a self-adjoint polynomial,*
- $f \in \mathcal{C}^{4k+7}(\mathbb{R})$.

Then there exist determinitic constants $(\alpha_i^P(f))_{i\in\mathbb{N}}$ *such that,*

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}_N\left(f(P(U^N))\right)\right] = \sum_{0\le i\le k}\frac{1}{N^{2i}}\alpha_i^P(f) + \mathcal{O}(N^{-2(k+1)}).$$

Besides, if the support of f *and the spectrum of* $P(u)$ *are disjoint, then for any* i, $\alpha_i^P(f) = 0$.

Idea of the proof

We want to show the following formula:

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}_N\left(f(P(U^N))\right)\right] = \sum_{0\leq i\leq k}\frac{1}{N^{2i}}\alpha_i^P(f) + \mathcal{O}(N^{-2(k+1)}).$$

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- We set v_s a family of free unitary Browian motions started in 1,

$$\mathbb{E}\left[\tau_N\Big(Q(U^N, v_s)\Big)\right] = \tau\Big(Q(u, v_s)\Big) - \int_0^\infty \mathbb{E}\left[\tau_N\Big(\mathcal{L}Q(u_t^N, v_s)\Big)\right] dt.$$

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- Then we show that there is a deterministic operator $T_{t,s}$ on the space of polynomials such that

$$\mathbb{E}\left[\tau_N\Big(\mathcal{L}Q(u_t^N, v_s)\Big)\right] = \frac{1}{N^2}\mathbb{E}\left[\tau_N\Big(T_{t,s}(Q)\big(u_t^N, v_s\big)\Big)\right].$$

- We can view the polynomial $T_{t,s}(Q)$ as a polynomial in (U^N, u_t, v_s) and reiterate the process.

Strong convergence of polynomial with matricial coefficients

Corollary

We define,

- $U^N = (U_1^N, \dots, U_d^N)$ *independent Haar unitary matrices of size* N,
- $u = (u_1, \dots, u_d)$ *free Haar unitaries,*
- $P = \sum_i Q_i(U, U^*) \otimes Y_i^M$ *with* Q_i *a non-commutative polynomial with* $Y_i^M \in \mathbb{M}_M(\mathbb{C})$.

If we assume that the family Y^M *are uniformly bounded over* M *for the operator norm, then for any* $\delta > 0$,

$$\mathbb{P}\left(\left\| P\left(U^N\right) \right\| \geq \| P(u) \| + \delta + \mathcal{O}\left(\left(\frac{M}{N}\right)^{1/2} \ln(NM)^{5/4} \right) \right) \leq e^{-\delta^2 N}.$$

Thus, if $M \ll N/\ln^{5/2}(N)$ *and that a family* Z^M *converges strongly in distribution towards a family of non-commutative variable* z, *then the family* $(U^N, \otimes I_M, I_N \otimes Z^M)$ *also converges strongly towards* $(u \otimes 1, 1 \otimes z)$.

Idea of the proof: the moment method

- Given $n \in \mathbb{N}$, $Q = P^*P$ a self-adjoint polynomial, one has proved that for some operator Δ,

$$\mathbb{E}\left[\mathrm{Tr}\left(Q^n\left(U^N\right)\right)\right] = NM\ \tau(P(u)) + \frac{1}{N^2}\mathbb{E}\left[\mathrm{Tr}\left((\Delta Q^n)\left(U^N\right)\right)\right].$$

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- One can write

$$\Delta Q^n(U^N) = \sum_{i+j+k+l=n-4} \alpha\left(Q^i\left(U^N\right), Q^j\left(U^N\right), Q^k\left(U^N\right), Q^l\left(U^N\right)\right),$$

where

$$\alpha(A_1 \otimes B_1, A_2 \otimes B_2, A_3 \otimes B_3, A_4 \otimes B_4) = A_2A_1A_4A_3 \otimes B_1B_2B_3B_4.$$

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- Thus

$$\begin{aligned}\mathbb{E}\left[\mathrm{Tr}\left(Q^n\left(U^N\right)\right)\right] &\leq NM\left\|P(u)\right\|^n + \frac{Cn^4M^2}{N^2}\mathbb{E}\left[\mathrm{Tr}\left(Q^{n-4}\left(U^N\right)\right)\right] \\ &\leq NM\left\|P(u)\right\|^n \times \frac{1}{1-\frac{C'n^4M^2}{N^2}}.\end{aligned}$$

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- Finally

$$\mathbb{E}\left[\left\|Q\left(U^N\right)\right\|\right] \leq \mathbb{E}\left[\mathrm{Tr}\left(Q^n\left(U^N\right)\right)\right]^{1/n} \leq \|P(u)\|\,(1+o(1)).$$

Problem

How do you compute the following quantity:

$$X = \int_{\mathbb{U}_N} U_{i_1,j_1} \dots U_{i_d,j_d} \overline{U}_{i'_1,j'_1} \dots \overline{U}_{i'_d,j'_d} dU,$$

where the integral is with respect to the Haar measure.

A link with Weingarten calculus

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where the integral is with respect to the Haar measure.

Given a Haar unitary matrix U of size N, one has

$$X = \mathbb{E}\left[\mathrm{Tr}_N\left(UE_{j_1,i_2}U \dots E_{j_{d-1},i_d}UE_{j_d i'_1}U^* E_{j'_1,i'_2} \dots U^* E_{j'_d,i_1}\right)\right].$$

Thus one wants to compute

$$N\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left(P(U, U^*, Z^N)\right)\right],$$

with $Z^N = (E_{i,j})_{i,j\in[1,N]}$.

A link with Weingarten calculus

However with $\tau_N = N^{-1}\,\mathrm{Tr}$, in the process of proving the asymptotic expansion, we have proved that there exist an operator Δ such that

$$\mathbb{E}\left[\tau_N\Big(P\left(U^N, Z^N\right)\Big)\right] = \tau(P(u, Z^N)) + \frac{1}{N^2}\mathbb{E}\left[\tau_N\Big((\Delta P)\left(U^N, Z^N\right)\Big)\right],$$

that is,

$$\mathbb{E}\left[\tau_N\left(\left(\mathrm{id} - \frac{1}{N^2}\Delta\right)(P)\left(U^N, Z^N\right)\right)\right] = \tau\left(P\left(u, Z^N\right)\right).$$

Consequently,

$$\mathbb{E}\left[\tau_N\left(P\left(U^N, Z^N\right)\right)\right] = \tau\left(\left(\mathrm{id} - \frac{1}{N^2}\Delta\right)^{-1}(P)\left(u, Z^N\right)\right).$$