On second order cumulants

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joint work with James Mingo

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Review: First order R-diagonal and even elements.
Definition

A non commutative probability space is a pair \((\mathcal{A}, \tau)\), where \(\mathcal{A}\) is an algebra with unity \(1_{\mathcal{A}}\) and \(\tau\) is a positive linear functional such that \(\tau(1_{\mathcal{A}}) = 1\).

We assume in this talk that \(\mathcal{A}\) is a \(C^*\)-algebra and \(\tau\) is tracial.

Definition (Voiculescu)

Let \((\mathcal{A}, \tau)\) be a NCPS and let \(\mathcal{A}_1, \mathcal{B}_2\) be subalgebras of \(\mathcal{A}\). We say that \(\mathcal{A}_1\) is free from \(\mathcal{B}_2\) if for all \(a_1 \in \mathcal{A}_1, b_i \in \mathcal{B}_2\) such that \(\tau(a_1) = 0 = \tau(b_i) = 0\) then \(\tau(a_1 b_1 a_2 b_2 \cdots a_n b_n) = 0\).
(First order) Non Commutative Probability

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**Definition (Voiculescu)**

Let \((\mathcal{A}, \tau)\) be a NCPS and let \(A_1, B_2\) be subalgebras of \(A\). We say that \(A_1\) is free from \(B_2\) if for all \(a_i \in A_1\) \(b_i \in B_2\) such that \(\tau(a_1) = 0 = \tau(b_i) = 0\) then

\[\tau(a_1 b_1 a_2 b_2 \cdots a_n b_n) = 0.\]
Definition (Convergence in joint distribution)

Let \((A_n, \tau_n)\) and \((A, \tau)\) be NCPS and let \(a_n, b_n \in A_n\). We say that the pair \((a_n, b_n)\) converges in joint distribution to \((a, b)\)

\[
\tau_n(a_1^{l_1} b_1^{m_1} \cdots a_k^{l_k} b_k^{m_k}) \to \tau(a_1^{l_1} b_1^{m_1} \cdots a_k^{l_k} b_k^{m_k}).
\]
Definition (Convergence in joint distribution)

Let \((A_n, \tau_n)_{n>0}\) and \((A, \tau)\) be NCPS and let \(a_n, b_n \in A_n\). We say that the pair \((a_n, b_n)\) converges in joint distribution to \((a, b)\)

\[
\tau_n(a_{n1}^{l1} b_{n1}^{m1} \cdots a_{nk}^{lk} b_{nk}^{mk}) \rightarrow \tau(a^{l1} b^{m1} \cdots a^{lk} b^{mk}).
\]

Two sequences of random variables \(\{a_n\}_n\) and \(\{b_n\}_n\) are said to be asymptotically free if they converge to a pair \(a, b\) of free random variables in some NCPS.
Theorem

Let \( \{ G_1^{(n)} \}_{n>0} \) and \( \{ G_2^{(n)} \}_{n>0} \) be two independent sequence of \( n \times n \) hermitian Gaussian matrices, then \( \{ G_1^{(n)} \}_{n>0} \) and \( \{ G_2^{(n)} \}_{n>0} \) are asymptotically free, as \( n \to \infty \).

Let \( \{ A_n \}_{n>0} \) and \( \{ B_n \}_{n>0} \) be two sequences of deterministic random matrices with limiting distributions and let \( U_n \) be a Haar distributed unitary matrix. Then \( \{ U_n A U_n^* \}_{n>0} \) and \( \{ B_n \}_{n>0} \) are asymptotically free.
Let $G_a : \mathbb{C}^+ \to \mathbb{C}^-$ be the Cauchy transform of $\mu \in \mathcal{M}$

$$G_a(z) = \tau \left( \frac{1}{z - a} \right).$$

The $R$-transform is given by $R_\mu(z) = G_\mu^{<-1>}(z) - 1/z$,

The free cumulants are the coefficients $\kappa_n = \kappa_n(a)$ in the series

$$R_a(z) = \sum_{n=1}^{\infty} k_{n+1} z^n.$$

They may be defined by the moment-cumulant formula

$$\tau(a^n) = \sum_{\sigma \in NC(n)} k_\sigma$$
Multivariate cumulants: multilinear functional $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ defined by

$$\tau(a_1 \cdots a_n) = \sum_{\sigma \in \mathcal{NC}(n)} \kappa_\sigma(a_1, \ldots, a_n)$$

$$\kappa_3(a_1, a_6, a_7) \quad \kappa_3(a_8, a_{11}, a_{12})$$

$$\kappa_1(a_9) \kappa_1(a_{10}) \quad \kappa_4(a_1, a_3, a_4, a_5)$$
Theorem (Speicher)

The following are equivalent

1. $A$ and $B$ are free.
2. For $a_i \in A$ and $b_j \in B$ mixed cumulants vanish:

$$k_n(\ldots, a_i, \ldots, b_k, \ldots) = 0 \quad (1)$$
Theorem (Krackwick & Speicher 00)

Let $n_1, \ldots, n_r$ be positive integers and $n = n_1 + \cdots + n_r$. Let $a_1 \ldots, a_n \in (\mathcal{A}, \tau)$. Then

$$
\kappa_r(a_1 \cdots a_{n_1}, \ldots, a_{n_1+\cdots+n_{r-1}+1} \cdots a_{n_1+\cdots+n_r}) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \ldots, a_n)
$$

where the sum is over all $\pi$’s such that $\pi \vee \tau_{\bar{n}} = 1_n$ and $\tau_{\bar{n}}$ is the partition $\{\{1, \ldots, n_1\}, \ldots, \{n_1 + \cdots + n_{r-1} + 1, \ldots, n_1 + \cdots + n_r\}\}$. 
Example: \(\{a_1, a_3\}\) free from \(\{a_2, a_4\}\)

\[
\kappa_2(a_1a_2, a_3a_4) = \kappa_4(a_1, a_2, a_3, a_4) + \kappa_1(a_1)\kappa_3(a_2, a_3, a_4) \\
+ \kappa_3(a_1, a_3, a_4)\kappa_1(a_2) + \kappa_3(a_1, a_2, a_4)\kappa_1(a_3) + \kappa_3(a_1, a_2, a_3)\kappa_1(a_4) \\
+ \kappa_2(a_1, a_4)\kappa_2(a_2, a_3) + \kappa_2(a_1, a_4)\kappa_1(a_2)\kappa_1(a_3) \\
+ \kappa_1(a_1)\kappa_2(a_2, a_3)\kappa_1(a_4) + \kappa_2(a_1, a_3)\kappa_1(a_2)\kappa_1(a_4) \\
+ \kappa_1(a_1)\kappa_2(a_2, a_4)\kappa_1(a_3).
\]
Example: \( \{a_1, a_3\} \) free from \( \{a_2, a_4\} \)

\[
\kappa_2(a_1a_2, a_3a_4) = \kappa_4(a_1, a_2, a_3, a_4) + \kappa_1(a_1)\kappa_3(a_2, a_3, a_4) \\
+ \kappa_3(a_1, a_3, a_4)\kappa_1(a_2) + \kappa_3(a_1, a_2, a_4)\kappa_1(a_3) + \kappa_3(a_1, a_2, a_3)\kappa_1(a_4) \\
+ \kappa_2(a_1, a_4)\kappa_2(a_2, a_3) + \kappa_2(a_1, a_4)\kappa_1(a_2)\kappa_1(a_3) \\
+ \kappa_1(a_1)\kappa_2(a_2, a_3)\kappa_1(a_4) + \kappa_2(a_1, a_3)\kappa_1(a_2)\kappa_1(a_4) \\
+ \kappa_1(a_1)\kappa_2(a_2, a_4)\kappa_1(a_3).
\]
Example: \( \{a_1, a_3\} \) free from \( \{a_2, a_4\} \)

\[
\kappa_2(a_1a_2, a_3a_4) = \kappa_2(a_1, a_4)\kappa_1(a_2)\kappa_1(a_3) \\
+ \kappa_1(a_1)\kappa_2(a_2, a_4)\kappa_1(a_3).
\]

\[a_1\]
\[a_2\]
\[a_3\]
\[a_4\]
Theorem

Let $a, b \in \mathcal{A}$ be free random variables. Then

$$\tau(ab^n) = \sum_{\pi \in NC(n)} \kappa_\pi(a, \ldots, a) \tau_{Kr}(\pi)(b, \ldots, b)$$  \hspace{1cm} (2)

$$\kappa_{ab}^n = \sum_{\pi \in NC(n)} \kappa_\pi(a, \ldots, a) \kappa_{Kr}(\pi)(b, \ldots, b)$$  \hspace{1cm} (3)
R-diagonal and even elements

**Definition**

Let $a \in A$. We say that $a$ is *R-diagonal* if for every $n$ and every $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ we have that

$$\kappa_n(a^{(\epsilon_1)}, a^{(\epsilon_2)}, \ldots, a^{(\epsilon_n)}) = 0$$

whenever there is $1 \leq i < n$ with $\epsilon_i = \epsilon_{i+1}$. i.e., *-cumulants are zero except possibly $\kappa_{2l}(a, a^*, \ldots, a, a^*)$ and $\kappa_{2l}(a^*, a, \ldots, a^*, a)$.

**Definition**

Let $a \in A$ we say that a self-adjoint element $a$ is *even* if all its odd free cumulants vanish, i.e. $\kappa_{2l+1}(a, \ldots, a) = 0$ for $l \geq 0$.

**Examples to have in mind.** 1) Circular and Semicircular operators. 2) Haar unitary and Bernoulli
Theorem (Nica & Speicher 97)

Let $x = x^* \in (\mathcal{A}, \tau)$ be even. Then the free cumulants of $x^2$ can be calculated from the free cumulants of $x$ as follows.

$$\kappa_n(x^2, \ldots, x^2) = \sum_{\pi \in NC(n)} \alpha_\pi$$

where $\alpha_n(x) = k_{2n}(x, \ldots, x)$.

Theorem (Nica & Speicher 97)

Let $a \in \mathcal{A}$ an R-diagonal operator. Then

$$\kappa_n(a^* a, \ldots, a^* a) = \sum_{\pi \in NC(n)} \beta_\pi(a).$$

$\beta_n(a) = k_{2n}(a^*, a, a^*, \ldots, a^*, a)$. 

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R-diagonal and even elements

Idea of the proof: Use formula for products and arguments.
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\[ \hat{\pi} \in NC(n) \]
Idea of the proof: Use formula for products and arguments.

\[ \pi \in NC^2(2n) \]
Theorem

Let $u$ and $b$ be operators be such that $u$ is a Haar unitary and such that $u$ and $b$ are $*$-free then $ub$ is a $R$-diagonal.

Theorem

Let $a$ be second order $R$-diagonal and consider the off-diagonal matrix,

$$A := \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$$

as an element in $(M_2(\mathcal{A}), tr_2 \otimes \tau)$. Then

1. $A$ is even element with the same determining sequence as $a$: $\alpha_n^{(A)} = \beta_n^{(a)}$ for all $n$.
2. If $a$ is a selfadjoint even element and $u$ is a Haar unitary which is $*$-free from $a$, then $ua$ is a tracial $R$-diagonal element with the same determining sequence as $a$. 
Fluctuation Moments and Second Order Cumulants.
Let \((A_N)_{N \in \mathbb{N}}\) be an ensemble of random matrices and suppose that \(A_N\) has a limiting distribution \(\mu_A\) in the sense that for all integers \(p\)

\[
\alpha_p := \lim_{N} \frac{1}{N} E(\text{Tr}(A_N^p))
\]

exists and \(\alpha_p = \int x^p d\mu(x)\).

We are further interested in \(\text{Cov}(\text{Tr}(A_N^p), \text{Tr}(A_N^q))\).

Thus, if \(Y_{N,p} = \text{Tr}(A_N^p - \alpha_p I_N)\), if the limit

\[
\alpha_{p,q} := \lim_{N} E(Y_{N,p} Y_{N,q})
\]

exists, and for all \(r > 2\) and \(p_1, p_2, \ldots, p_r\)

\[
\lim_{N} c_r(\text{Tr}(A_N^{p_1}), \ldots, \text{Tr}(A_N^{p_r})) = 0,
\]

\((\alpha_{p,q})_{p,q}=\text{fluctuation moments}\) of the limiting distribution.
We want then to describe $\tau_2(\cdot, \cdot) = \lim \text{Cov}(\cdot, \cdot)$
The framework that we use is a second order probability space.

**Definition**

A *second order probability space* is a triplet $(\mathcal{A}, \tau, \tau_2)$ such that

- $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\tau : \mathcal{A} \to \mathbb{C}$ is a tracial linear functional with $\tau(1) = 1$.

- We assume that $\tau_2 : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is a bilinear, symmetric, tracial in each variable, and is such that $\tau_2(1, a) = \tau_2(a, 1) = 0$ for all $a \in \mathcal{A}$. 

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On second order cumulants
Second order cumulants

\[ \tau_2(a_1 \cdots a_m, a_{m+1} \cdots a_{m+n}) = \sum_{\pi \in S_{NC}(m,n)} \kappa_{\pi}(a_1, \ldots, a_{m+n}) + \sum_{(U, \pi) \in P S_{NC}(m,n)'} \kappa(U, \pi)(a_1, \ldots, a_{m+n}). \]
Second order cumulants

\[ \tau_2(a_1 \cdots a_m, a_{m+1} \cdots a_{m+n}) = \sum_{\pi \in S_{NC}(m,n)} \kappa_{\pi}(a_1, \ldots, a_{m+n}) + \sum_{(U, \pi) \in P S_{NC}(m,n)'} \kappa(U, \pi)(a_1, \ldots, a_{m+n}). \]

Left: at least 1 through block. Right: No through block.
Second order cumulants

\[ \tau_2(a_1 \cdots a_m, a_{m+1} \cdots a_{m+n}) = \sum_{\pi \in S_{NC}(m,n)} \kappa_{\pi}(a_1, \ldots, a_{m+n}) + \sum_{(U,\pi) \in PS_{NC}(m,n)'} \kappa(U,\pi)(a_1, \cdots, a_{m+n}). \]

\[ \alpha_{1,1} = \kappa_{1,1} + \kappa_2 \]
\[ \alpha_{2,1} = \kappa_{1,2} + 2\kappa_1 \kappa_{1,1} + 2\kappa_3 + 2\kappa_1 \kappa_2 \]
\[ \alpha_{2,2} = \kappa_{2,2} + 4\kappa_1 \kappa_{2,1} + 4\kappa_1^2 \kappa_{1,1} + 4\kappa_4 + 8\kappa_1 \kappa_3 + 2\kappa_2^2 + 4\kappa_1^2 \kappa_2 \]
\[ \alpha_{1,3} = \kappa_{1,3} + 3\kappa_1 \kappa_{2,1} + 3\kappa_2 \kappa_{1,1} + 3\kappa_1^2 \kappa_{1,1} + 3\kappa_4 + 6\kappa_1 \kappa_3 + 3\kappa_2^2 + 3\kappa_1^2 \kappa_2 \]
\[ \alpha_{2,3} = \kappa_{2,3} + 2\kappa_1 \kappa_{1,3} + 3\kappa_1 \kappa_{2,2} + 3\kappa_2 \kappa_{1,2} + 9\kappa_1^2 \kappa_{1,2} + 6\kappa_1 \kappa_2 \kappa_{1,1} + 6\kappa_1^3 \kappa_{1,1} \]
\[ + 6\kappa_5 + 18\kappa_1 \kappa_4 + 12\kappa_2 \kappa_3 + 18\kappa_1^2 \kappa_3 + 12\kappa_1 \kappa_2^2 + 6\kappa_1^3 \kappa_2 \]
Second order cumulants series

First and second order cumulant series:

\[ R(z) = \frac{1}{z} \sum_{n \geq 1} \kappa_n^a z^n, \quad R(z, w) = \frac{1}{zw} \sum_{p, q \geq 1} \kappa_{p,q}^a z^p w^q \]

where \( \kappa_n^a = \kappa_n(a, \ldots, a) \) and \( \kappa_{p,q}^a = \kappa_{p,q}(a, \ldots, a) \).

Moment series,

\[ G(z) = \frac{1}{z} \sum_{n \geq 0} \tau(a^n) z^{-n}, \quad G(z, w) = \frac{1}{zw} \sum_{n, m \geq 1} \tau(a^n, a^m) z^{-n} w^{-m}. \]

Then we have the relations

\[ \frac{1}{G(z)} + R(G(z)) = z \]

and

\[ G(z, w) = G'(z) G'(w) R(G(z), G(w)) + \frac{\partial^2}{\partial z \partial w} \log \left( \frac{G(z) - G(w)}{z - w} \right). \]
Second order freeness

Definition

The random variables \( a_1, \ldots, a_n \) are second order free if

- They are free.
- The second order \textit{mixed} cumulants vanish.
The random variables $a_1, \ldots, a_n$ are second order free if

- They are free.
- The second order mixed cumulants vanish.

Matrices are asymptotically second order free if the first and second order mixed cumulants vanish in the limit.

Examples. Complex Gaussian matrices, and Wishart, $(UAU^*, B)$ with $A$ and $B$ deterministic.

Non examples: General Wigner matrices.
Theorem (Mingo Speicher Tan 08)

Suppose \( n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+s} \) are positive integers, \( p = n_1 + \cdots + n_r, q = n_{r+1} + \cdots + n_{r+s} \), and

\[
N = \{ n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{r+s} \}
\]

Given a second probability space \((A, \tau, \tau_2)\) and

\[
a_1, \ldots, a_{n_1}, a_{n_1+1}, \ldots, a_{n_1+n_2}, \ldots, a_n + \cdots + n_{r+s} \in A
\]

let \( A_1 = a_1 \cdots a_{n_1}, A_2 = a_{n_1+1} \cdots a_{n_1+n_2}, \ldots, A_{r+s} = a_{n_1+\cdots+n_{r+s-1}+1} \cdots a_{n_1+\cdots+n_{r+s}} \). Then

\[
\kappa_{r,s}(A_1, \ldots, A_r, A_{r+1}, \ldots, A_{r+s}) = \sum_{(V, \pi)} \kappa(V, \pi)(a_1, \ldots, a_{p+q}), \quad (4)
\]

where the summation is over those \((V, \pi) \in \mathcal{PS}_{NC}(p, q)\) such that \(\pi^{-1}\gamma_{p,q}\) separates the points of \(N\).
Formula for products as arguments

Separating the points
Second Order diagonal and Even elements
R-diagonal elements

**Definition**

Let \((A, \tau, \tau_2)\) be a second order non-commutative probability space. An element \(a \in (A, \tau, \tau_2)\) is called **second order R-diagonal** if it is \(R\)-diagonal and the only non-vanishing second order cumulants are of the form

\[
\kappa_{2p,2q}(a, a^*, \ldots, a, a^*) = \kappa_{2p,2q}(a^*, a, \ldots, a^*, a).
\]

**Examples:**

- Second order Haar Unitary: Limiting distribution of Haar distributed on \(U(n)\). (not obvious)
- Circular operator \(c\): Limiting distribution of \(G_1 + iG_2\), GUE matrices.
Definition

Let \((A, \tau, \tau_2)\) be a second order non-commutative probability space. An element \(x \in (A, \tau, \tau_2)\) is called **second order even** (or even for short) if \(x = x^*\) and \(x\) is such that odd moments vanish, i.e. \(\tau_2(x^p, x^q) = 0\) unless \(p\) and \(q\) are even and \(\tau(x^{2n+1}) = 0\) for all \(n \geq 0\).

Example:

- Semicircular operator \(c\): Limiting distribution of \(G\) a GUE matrices.
Theorem

Let \( a \) be second order \( R \)-diagonal in \((A, \tau, \tau_2)\) and consider the off-diagonal matrix,

\[
A := \begin{pmatrix}
0 & a \\
\ast a & 0
\end{pmatrix}
\]
as an element in \((M_2(A), tr_2 \otimes \tau, \tilde{\tau}_2)\). Then

1. \( A \) is second order even element.

2. \( A \) has the same determining sequence as \( a \): \( \beta^{(A)}_{p,q} = \beta^{(a)}_{p,q} \) for all \( p, q \geq 1 \) and \( \beta^{(A)}_n = \beta^{(a)}_n \) for all \( n \).

3. The second order cumulants of \( a \) and the second order cumulants of \( A \), are related by

\[
\kappa^{(a)}_{p,q} = \kappa^{(A)}_{p,q} + \sum_{\pi \in S_{NC}^{all,+}(m,n)} \kappa^{(A)}_{\pi}.
\]
Theorem

Let \( \{a, a^*\} \) and \( \{b, b^*\} \) be second order free and suppose that \( a \) is second order \( R \)-diagonal. Then \( ab \) is second order \( R \)-diagonal.

Examples:

- \( c_1 c_2 \cdots c_n, c_i \) second order Circular Operator.
- \( ua, u \) second order Haar, a limit of \( A_n \) deterministic.
Definition

Let \((A, \tau, \tau_2)\) be a second order non-commutative probability space. Let \(a\) be second order \(R\)-diagonal. Define 
\[ \beta_n^{(a)} := \kappa_{2n}(a, a^*, \ldots, a, a^*) \]
and 
\[ \beta_{p,q}^{(a)} := \kappa_{2p,2q}(a, a^*, \ldots, a, a^*). \] (5)

Theorem

Let \(a\) be a second order \(R\)-diagonal with determining sequences 
\((\beta_n^{(a)})_{n \geq 1}\) and \((\beta_{p,q}^{(a)})_{p,q \geq 1}\) then we have 
\[ \kappa_{p,q}(aa^*, \ldots, aa^*) = \sum_{(\mathcal{V},\pi) \in \mathcal{PS}_{NC}(p,q)} \beta_{(\mathcal{V},\pi)}. \] (6)
\[ \kappa_{p,q}(aa^*, \ldots, aa^*) = \sum_{(\nu,\pi) \in \text{PS}_{NC}(p,q)} \beta_{\nu,\pi}(a) \].

**Figure:** On the left we see the permutation \( \pi = (1, 14, 15, 12)(2, 3)(4, 5, 18, 13)(6, 7)(8, 9, 10, 11, 16, 17) \). \( \pi \) is in \( S_{NC}^- (12, 6) \) and \( \gamma_{12,6}(1, \pi)^{-1} \) separates the points of \( O = \{1, 3, 5, 7, 9, 11, 13, 15, 17\} \). On the right is \( \tilde{\pi} = (1)(2, 9)(3)(4, 5, 8)(6, 7) \). to the even numbers.
Definition

Let \((A, \tau, \tau_2)\) be a second order non-commutative probability space. Let \(x\) be a second order even element. Define
\[
\beta_n^{(x)} := \kappa_{2n}^{(x)} := \kappa_{2n}(x, \ldots, x)
\]
and letting \(\kappa_{p, q}^{(x)} = \kappa_{p, q}(x, \ldots, x)\), we set
\[
\beta_{p, q}^{(x)} := \kappa_{2p, 2q}^{(x)} + \sum_{\pi \in S_{NC}^{all, +}(2p, 2q)} \kappa_{p, q}^{(x)}
\]  
(7)

Theorem

Let \(x\) be an even element with determining sequences \((\beta_n^{(x)})_{n \geq 1}\) and \((\beta_{p, q}^{(x)})_{p, q \geq 1}\) then the second order cumulants of \(x^2\) are given by
\[
\kappa_{p, q}(x^2, \ldots, x^2) = \sum_{(\nu, \pi) \in PS_{NC}(p, q)} \beta_{\nu, \pi}^{(x)}.
\]  
(8)
Main difference in using Mingo-Speicher-Tan formula:
Main difference in using Mingo-Speicher-Tan formula:
Main difference in using Mingo-Speicher-Tan formula:
Theorem

Let $x$ be either second order even or second order $R$-diagonal element with determining sequences $(\beta_n)_{n \geq 1}$ and $(\beta_{p,q})_{p,q \geq 1}$, define the formal power series

\[ B(z) = \frac{1}{z} \sum_{n \geq 1} \beta_n z^n, \quad B(z, w) = \frac{1}{zw} \sum_{p,q \geq 1} \beta_{p,q} z^p w^q, \]

\[ C(z) = \frac{1}{z} + \frac{1}{z} \sum_{n \geq 1} \kappa_{xx}^* z^{-n}, \quad C(z, w) = \frac{1}{zw} \sum_{n,m \geq 1} \kappa_{p,q}^{xx} z^{-p} w^{-q}. \]

Then

\[ \frac{1}{C(z)} + B(C(z)) = z \]

and

\[ C(z, w) = C'(z) C'(w) B(C(z), C(w)) + \frac{\partial^2}{\partial z \partial w} \log \left( \frac{C(z) - C(w)}{z - w} \right). \]
Examples
s is called a second order semi-circular operator if its first order cumulants satisfy $\kappa_n(s, s, \ldots, s) = 0$ for all $n \neq 2$ and $\kappa_2(s, s) = 1$, and for all $p$ and $q$ the second order cumulants $\kappa_{p,q}$ are 0. This operator appears as the limit of GUE random matrices as the size tends to infinity.

Consider $s_1$ and $s_2$ second order free semicircular operators. We call $c = \frac{s_1 + is_2}{\sqrt{2}}$ a (second order) circular operator. The operator $c$ is a second order $R$-diagonal. Indeed, since $s_1$ and $s_2$ are second order free, their mixed free cumulants vanish, also the second order cumulant of $s_1$ and $s_2$ vanish thus by linearity the same is true for $c$. That is, $\kappa_{p,q}(c(\epsilon_1), \ldots, c(\epsilon_{p+q})) = 0$ for all $\epsilon_1, \ldots, \epsilon_{p+q} \in \{\pm 1\}$.
The determining sequence of a semicircle operator is given by $\beta_1 = 1$ and $\beta_n = 0$ for $n > 0$. The second order determining sequence is given by $\beta_{k,k} = k$ and $\beta_{p,q} = 0$ if $p \neq q$. Thus second order cumulants of $s^2$.

\[
\kappa_{p,q}(s^2, \ldots, s^2) = \sum_{\pi \in S_{NC}(p,q)} \beta_\pi + \sum_{(\nu,\pi) \in \mathcal{P} S_{NC}(p,q)'} \beta_{(\nu,\pi)} = \sum_{\pi \in \mathcal{P} S_{NC}(p,q)'} \beta_{(\nu,\pi)}
\]

\[
= \sum_{k > 0} \binom{p}{k} \binom{q}{k} \beta_{k,k} = \sum_{k > 0} k \binom{p}{k} \binom{q}{k} = p \binom{p + q - 1}{p}.
\]
Recall that $cc^*$ and $s^2$ both have a free Poisson distribution with respect to $\tau$ (i.e. $\tau\left((cc^*)^n\right) = \tau\left((s^2)^n\right) = \frac{1}{n+1}\binom{2n}{n}$). So one might expect that $cc^*$ and $s^2$ have the same distribution in the second order level. We see that this is not the case.

Indeed, the determining sequences of $c$ are given by $\beta_1 = 1$ and $\beta_n = 0$ for $n > 0$ and $\beta_{p,q} = 0$, for all $p$ and $q$. It is readily seen that the second order cumulants of $cc^*$ are all zero. i.e. $\kappa_{p,q}(cc^*, cc^*, ..., cc^*) = 0$.

Since the first order cumulants of $cc^*$ are all 1. Then the $(p, q)$-fluctuation moments of $cc^*$ count the number of elements in $S_{NC}(p, q)$.
Theorem (Second order Moments and Cumulants of Products of Free Variables)

Let $a_1, \ldots, a_k$ be operators which are second order free and such that $\kappa_{p,q}^{(a_i)} = 0$ for all $p$ and $q$. Denote by $a := a_1 a_2 \cdots a_k$.

$$\tau(a^p, a^q) = \sum_{\pi \in S_{NC}^{k-\text{alt}}(p,q)} \kappa_{Kr(\pi)}(a, \ldots, a). \quad (9)$$

Furthermore,

$$\kappa_{p,q}^{(a)} = \sum_{\pi \in S_{NC}^{k-\text{alt-eq}}(p,q)} \kappa_{Kr(\pi)}(a, \ldots, a). \quad (10)$$
Let $h = c_1 c_2 \cdots c_k$. Equations (9) and (10) of Theorem 26 are very useful. Indeed, a direct application of them gives a combinatorial description of the fluctuation moments and cumulants:

$$\tau_2(h^p, h^q) = |\text{SNC}_{\text{alt}}^k(kp, kq)| \quad \text{and} \quad \kappa_{p,q}(h, \ldots, h) = |\text{SNC}_{\text{alt}}^k(kp, kq)|,$$

a precise formula for this quantities or a generating function is not known for $k > 2$. 

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On second order cumulants
The $c$ be a 2nd order circular operator which is second order $\ast$-free from $a$. We are interested in the second order cumulants of $cac^*$. We can show that as in the first order level

$$\kappa_{p,q}(cac^*, \ldots, cac^*) = \sum_{(\nu,\pi) \in \mathcal{PS}_{NC}(p,q)} \kappa(\nu,\pi)(a, \ldots, a) = \tau_2(a^p, a^q).$$

A particular important example is the case when the fluctuation moments of $a$ are 0. In this case $G_a(z, w) = 0$ and the formula above is reduced to

$$G_{cac^*}(z, w) = \frac{\partial^2}{\partial z \partial w} \log \left( \frac{G_{cac^*}(z) - G_{cac^*}(w)}{z - w} \right).$$

From the random matrix perspective this corresponds to the case when $a$ is a limit of deterministic matrices and $cac^*$ then corresponds to $WAW^*$ where $W$ is a Ginibre matrix and $A$ is deterministic.
Two desirable properties for R-diagonal are not true in the second order level.

- If $r$ is 2nd order R diagonal $r^n$ is not necessarily 2nd order $R$-diagonal.
- If $u$ is 2nd order free from $a$. $ubu^*$ may not be 2nd order free from $a$. 

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On second order cumulants
Thanks!