On second order cumulants

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joint work with James Mingo

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Review: First order R-diagonal and even elements.

(First order) Non Commutative Probability

Definition

A non commutative probability space is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is an algebra with unity $1_{\mathcal{A}}$ and τ is a positive linear functional such that $\tau(1_{\mathcal{A}}) = 1$.

We assume in this talk that $\mathcal{A}$ is a C^*-algebra and τ is tracial.

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Definition (Voiculescu)

Let (A, τ) be a NCPS and let A_1, B_2 be subalgebras of A. We say that A_1 is free from B_2 if for all $a_i \in A_1$ $b_i \in B_2$ such that $\tau(a_1) = 0 = \tau(b_i) = 0$ then

$$\tau(a_1 b_1 a_2 b_2 \cdots a_n b_n) = 0.$$

Definition (Convergence in joint distribution)

Let $(A_n, \tau_n)_{n>0}$ and (A, τ) be NCPS an let $a_n, b_n \in A_n$. We say that the pair (a_n, b_n) converges in joint ditribution to (a, b)

$$\tau_n(a_n^{l_1} b_n^{m_1} \cdots a_n^{l_k} b_n^{m_k}) \to \tau(a^{l_1} b^{m_1} \cdots a^{l_k} b^{m_k}).$$

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- Two sequences of random variables $\{a_n\}_n$ and $\{b_n\}_n$ are said to be **asymptotically free** if they converge to a pair a, b of free random variables in some NCPS.

Asymptotic Freeness

Theorem

- *Let* $\{G_1^{(n)}\}_{n>0}$ *and* $\{G_2^{(n)}\}_{n>0}$ *be two independent sequence of* $n \times n$ *hermitian Gaussian matrices, then* $\{G_1^{(n)}\}_{n>0}$ *and* $\{G_2^{(n)}\}_{n>0}$ *are asymptotically free, as* $n \to \infty$.
- *Let* $\{A_n\}_{n>0}$ *and* $\{B_n\}_{n>0}$ *be two sequences of deterministic random matrices with limiting distributions and let* U_n *be a Haar distributed unitary matrix. Then* $\{U_n A U_n^*\}_{n>0}$ *and* $\{B_n\}_{n>0}$ *are asymptotically free.*

Free Cumulants

Let $G_a : \mathbb{C}^+ \to \mathbb{C}^-$ be the Cauchy transform of $\mu \in \mathcal{M}$

$$G_a(z) = \tau(\frac{1}{z-a}).$$

The R-transform is given by $R_\mu(z) = G_\mu^{<-1>}(z) - 1/z$,
The free cumulants are the coefficients $\kappa_n = \kappa_n(a)$ in the series

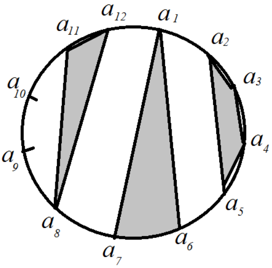
$$R_a(z) = \sum_{n=1}^{\infty} k_{n+1} z^n.$$

They may be defined by the moment-cumulant formula

$$\tau(a^n) = \sum_{\sigma \in \mathcal{NC}(n)} k_\sigma$$

Multivariate cumulants: multilinear functional $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ defined by

$$\tau(a_1 \cdots a_n) = \sum_{\sigma \in \mathcal{NC}(n)} \kappa_\sigma(a_1, \ldots, a_n)$$



$$\kappa_3(a_1, a_6, a_7)\ \kappa_3(a_8, a_{11}, a_{12})$$

$$\kappa_1(a_9)\kappa_1(a_{10})\ \kappa_4(a_1, a_3, a_4, a_5)$$

Free Cumulants

Theorem (Speicher)

The following are equivalent

- $\mathcal{A}$ *and* $\mathcal{B}$ *are free.*
- *For* $a_i \in \mathcal{A}$ *and* $b_j \in \mathcal{B}$ *mixed cumulants vanish:*

$$k_n(\ldots, a_l, \ldots, b_k, \ldots) = 0 \quad (1)$$

Formula for products as arguments

Theorem (Krackwick & Speicher 00)

Let $n_1, \ldots, n_r$ be positive integers and $n = n_1 + \cdots + n_r$. Let $a_1 \ldots, a_n \in (\mathcal{A}, \tau)$. Then

$$\kappa_r(a_1 \cdots a_{n_1}, \ldots, a_{n_1+\cdots+n_{r-1}+1} \cdots a_{n_1+\cdots+n_r}) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \ldots, a_n)$$

where the sum is over all π's such that $\pi \vee \tau_{\vec{n}} = 1_n$ and $\tau_{\vec{n}}$ is the partition $\{\{1, \ldots, n_1\}, \ldots, \{n_1 + \cdots + n_{r-1} + 1, \ldots, n_1 + \cdots + n_r\}$.

Formula for products as arguments

Example: $\{a_1, a_3\}$ free from $\{a_2, a_4\}$

$$\begin{aligned} \kappa_2(a_1a_2, a_3a_4) &= \kappa_4(a_1, a_2, a_3, a_4) + \kappa_1(a_1)\kappa_3(a_2, a_3, a_4) \\ + & \kappa_3(a_1, a_3, a_4)\kappa_1(a_2) + \kappa_3(a_1, a_2, a_4)\kappa_1(a_3) + \kappa_3(a_1, a_2, a_3)\kappa_1(a_4) \\ + & \kappa_2(a_1, a_4)\kappa_2(a_2, a_3) + \kappa_2(a_1, a_4)\kappa_1(a_2)\kappa_1(a_3) \\ + & \kappa_1(a_1)\kappa_2(a_2, a_3)\kappa_1(a_4) + \kappa_2(a_1, a_3)\kappa_1(a_2)\kappa_1(a_4) \\ + & \kappa_1(a_1)\kappa_2(a_2, a_4)\kappa_1(a_3). \end{aligned}$$

Formula for products as arguments

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$$
\begin{aligned}
& \kappa_2(a_1 a_2, a_3 a_4) = \kappa_4(a_1, a_2, a_3, a_4) + \kappa_1(a_1)\kappa_3(a_2, a_3, a_4) \\
+ \; & \kappa_3(a_1, a_3, a_4)\kappa_1(a_2) + \kappa_3(a_1, a_2, a_4)\kappa_1(a_3) + \kappa_3(a_1, a_2, a_3)\kappa_1(a_4) \\
+ \; & \kappa_2(a_1, a_4)\kappa_2(a_2, a_3) + \kappa_2(a_1, a_4)\kappa_1(a_2)\kappa_1(a_3) \\
+ \; & \kappa_1(a_1)\kappa_2(a_2, a_3)\kappa_1(a_4) + \kappa_2(a_1, a_3)\kappa_1(a_2)\kappa_1(a_4) \\
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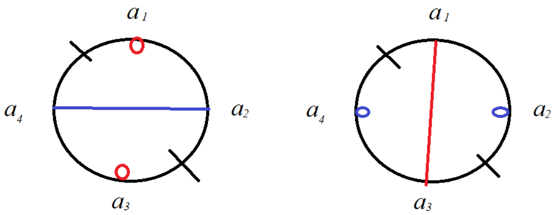
\end{aligned}
$$

Example: $\{a_1, a_3\}$ free from $\{a_2, a_4\}$

$$\kappa_2(a_1 a_2, a_3 a_4) = \kappa_2(a_1, a_4)\kappa_1(a_2)\kappa_1(a_3) + \kappa_1(a_1)\kappa_2(a_2, a_4)\kappa_1(a_3).$$



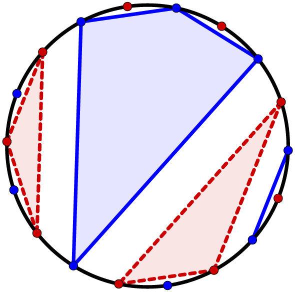
Formula for products as arguments

Theorem

Let $a, b \in \mathcal{A}$ be free random variables. Then

$$\tau(ab^n) = \sum_{\pi \in NC(n)} \kappa_\pi(a, \dots, a)\tau_{\mathrm{Kr}(\pi)}(b, \dots, b) \quad (2)$$

$$\kappa_n^{ab} = \sum_{\pi \in NC(n)} \kappa_\pi(a, \dots, a)\kappa_{\mathrm{Kr}(\pi)}(b, \dots, b) \quad (3)$$



R-diagonal and even elements

Definition

Let $a \in \mathcal{A}$. We say that a is *R-diagonal* if for every n and every $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ we have that

$$\kappa_n(a^{(\epsilon_1)}, a^{(\epsilon_2)}, \dots, a^{(\epsilon_n)}) = 0$$

whenever there is $1 \leq i < n$ with $\epsilon_i = \epsilon_{i+1}$. i.e., $*$-cumulants are zero except possibly $\kappa_{2l}(a, a^*, \dots, a, a^*)$ and $\kappa_{2l}(a^*, a, \dots, a^*, a)$.

Definition

Let $a \in \mathcal{A}$ we say that a self-adjoint element a is *even* if all its odd free cumulants vanish, i.e. $\kappa_{2l+1}(a, \dots, a) = 0$ for $l \geq 0$.

Examples to have in mind. 1) Circular and Semicircular operators. 2) Haar unitary and Bernoulli

R-diagonal and even elements

Theorem (Nica & Speicher 97)

Let $x = x^* \in (\mathcal{A}, \tau)$ be even. Then the free cumulants of x^2 can be calculated from the free cumulants of x as follows.

$$\kappa_n(x^2, \dots, x^2) = \sum_{\pi \in NC(n)} \alpha_\pi$$

where $\alpha_n(x) = k_{2n}(x, \dots, x)$.

Theorem (Nica & Speicher 97)

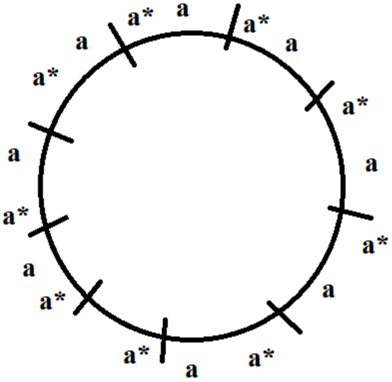
Let $a \in \mathcal{A}$ an R-diagonal operator. Then

$$\kappa_n(a^*a, \dots, a^*a) = \sum_{\pi \in NC(n)} \beta_\pi(a).$$

$$\beta_n(a) = k_{2n}(a^*, a, a^*, \dots, a^*, a).$$

R-diagonal and even elements

Idea of the proof: Use formula for products and arguments.



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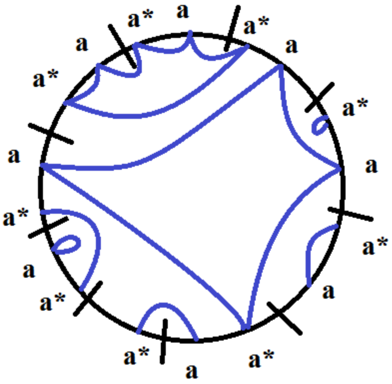

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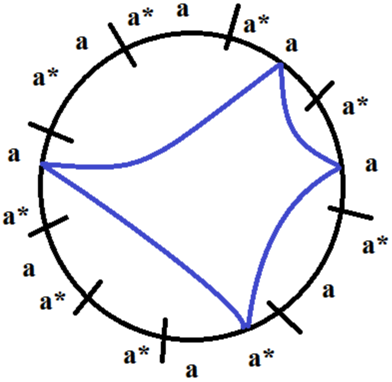

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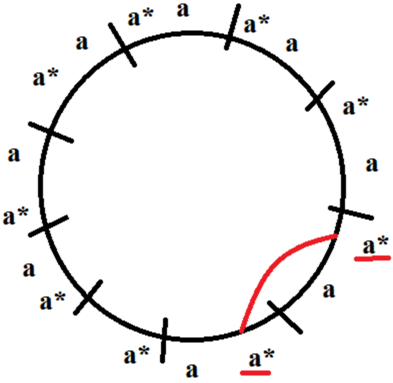

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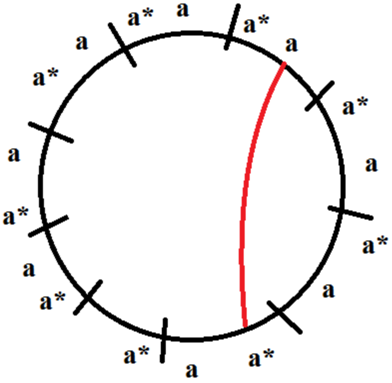

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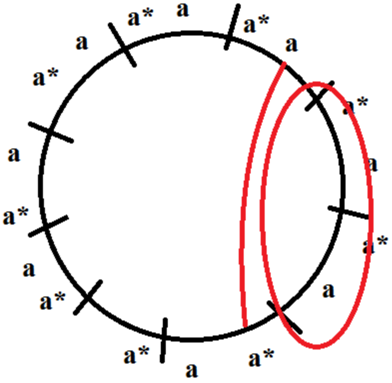

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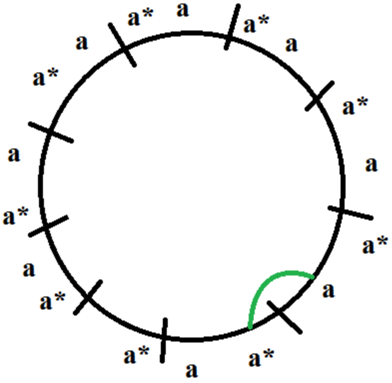

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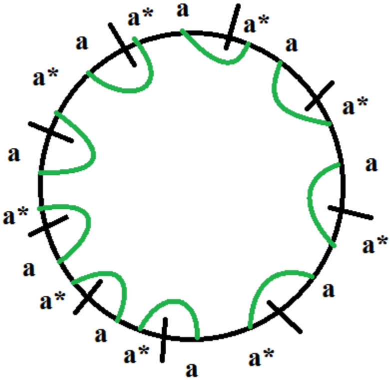

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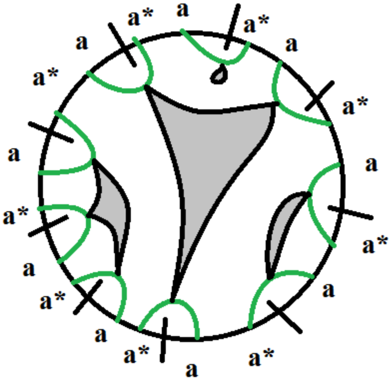
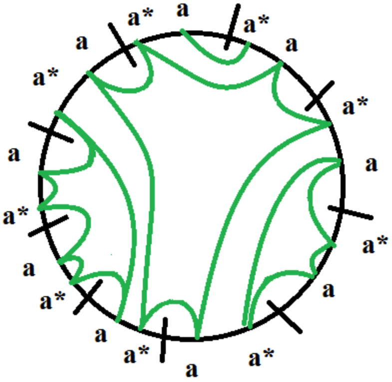


$$\hat{\pi} \in NC(n)$$

Idea of the proof: Use formula for products and arguments.



$$\pi \in NC^2(2n)$$

R-diagonal to even elements

Theorem

Let u and b be operators be such that u is a Haar unitary and such that u and b are $*$-free then ub is a R-diagonal.

Theorem

Let a be second order R-diagonal and consider the off-diagonal matrix,

$$A := \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$$

as an element in $(M_2(\mathcal{A}), tr_2 \otimes \tau)$. Then

1. *A is even element with the same determining sequence as a: $\alpha_n^{(A)} = \beta_n^{(a)}$ for all n.*
2. *If a is a selfadjoint even element and u is a Haar unitary which is $*$-free from a, then ua is a tracial R-diagonal element with the same determining sequence as a.*

Fluctuation Moments and Second Order Cumulants.

Fluctuation Moments

Let $(A_N)_{N\in\mathbb{N}}$ be an ensemble of random matrices and suppose that A_N has a limiting distribution μ_A in the sense that for all integers p

$$\alpha_p := \lim_N \frac{1}{N} E(\mathrm{Tr}(A_N^p))$$

exists and $\alpha_p = \int x^p d\mu(x)$.
We are further interested in $Cov(\mathrm{Tr}(A_N^p), \mathrm{Tr}(A_N^q)$.
Thus, if $Y_{N,p} = \mathrm{Tr}(A_N^p - \alpha_p I_N)$, if the limit

$$\alpha_{p,q} := \lim_N E(Y_{N,p} Y_{N,q})$$

exists, and for all $r > 2$ and $p_1, p_2, \ldots, p_r$

$$\lim_N c_r(\mathrm{Tr}(A_N^{p_1}), \ldots, \mathrm{Tr}(A_N^{p_r})) = 0,$$

$(\alpha_{p,q})_{p,q}$=*fluctuation moments* of the limiting distribution.

Second order probability space

We want then to describe $\tau_2(\cdot,\cdot) = \lim Cov(\cdot,\cdot)$
The framework that we use is a second order probability space.

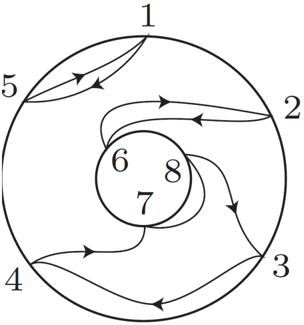
Definition

A *second order probability space* is a triplet $(\mathcal{A}, \tau, \tau_2)$ such that

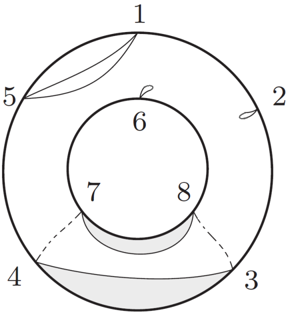
- $\mathcal{A}$ is aunital algebra over $\mathbb{C}$ and $\tau : \mathcal{A} \to \mathbb{C}$ is a tracial linear functional with $\tau(1) = 1$.
- We assume that $\tau_2 : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is a bilinear, symmetric, tracial in each variable, and is such that $\tau_2(1, a) = \tau_2(a, 1) = 0$ for all $a \in \mathcal{A}$.

Second order cumulants

$$\tau_2(a_1 \cdots a_m, a_{m+1} \cdots a_{m+n}) = \sum_{\pi \in S_{NC}(m,n)} \kappa_\pi(a_1, \ldots, a_{m+n}) + \sum_{(\mathcal{U},\pi) \in \mathcal{PS}_{NC}(m,n)'} \kappa_{(\mathcal{U},\pi)}(a_1, \cdots, a_{m+n}).$$



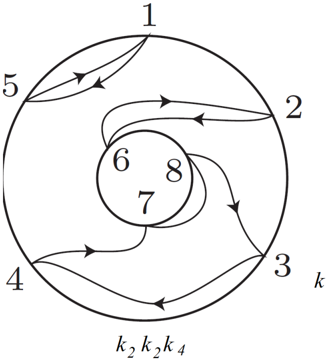
$S_{NC}(5,3)$

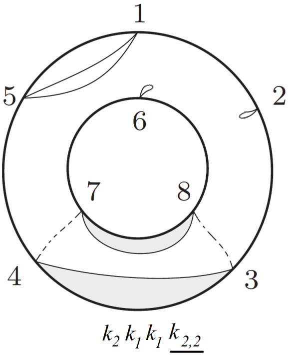


$PS'_{NC}(5,3)$

Second order cumulants

$$\tau_2(a_1 \cdots a_m, a_{m+1} \cdots a_{m+n})$$
$$= \sum_{\pi \in S_{NC}(m,n)} \kappa_\pi(a_1, \ldots, a_{m+n}) + \sum_{(\mathcal{U},\pi) \in \mathcal{PS}_{NC}(m,n)'} \kappa_{(\mathcal{U},\pi)}(a_1, \cdots, a_{m+n}).$$





Left: at least **1** through block. Right: No through block.

Second order cumulants

$$\tau_2(a_1 \cdots a_m, a_{m+1} \cdots a_{m+n})$$
$$= \sum_{\pi \in S_{NC}(m,n)} \kappa_\pi(a_1, \ldots, a_{m+n}) + \sum_{(\mathcal{U},\pi) \in \mathcal{PS}_{NC}(m,n)'} \kappa_{(\mathcal{U},\pi)}(a_1, \cdots, a_{m+n}).$$

$$\alpha_{1,1} = \kappa_{1,1} + \kappa_2$$
$$\alpha_{2,1} = \kappa_{1,2} + 2\kappa_1\kappa_{1,1} + 2\kappa_3 + 2\kappa_1\kappa_2$$
$$\alpha_{2,2} = \kappa_{2,2} + 4\kappa_1\kappa_{2,1} + 4\kappa_1^2\kappa_{1,1} + 4\kappa_4 + 8\kappa_1\kappa_3 + 2\kappa_2^2 + 4\kappa_1^2\kappa_2$$
$$\alpha_{1,3} = \kappa_{1,3} + 3\kappa_1\kappa_{2,1} + 3\kappa_2\kappa_{1,1} + 3\kappa_1^2\kappa_{1,1} + 3\kappa_4 + 6\kappa_1\kappa_3 + 3\kappa_2^2 + 3\kappa_1^2\kappa_2$$
$$\alpha_{2,3} = \kappa_{2,3} + 2\kappa_1\kappa_{1,3} + 3\kappa_1\kappa_{2,2} + 3\kappa_2\kappa_{1,2} + 9\kappa_1^2\kappa_{1,2} + 6\kappa_1\kappa_2\kappa_{1,1} + 6\kappa_1^3\kappa_{1,1}$$
$$+ 6\kappa_5 + 18\kappa_1\kappa_4 + 12\kappa_2\kappa_3 + 18\kappa_1^2\kappa_3 + 12\kappa_1\kappa_2^2 + 6\kappa_1^3\kappa_2$$

Second order cumulants series

First and second order cumulant series:

$$R(z) = \frac{1}{z}\sum_{n\geq 1}\kappa_n^a z^n, \qquad R(z,w) = \frac{1}{zw}\sum_{p,q\geq 1}\kappa_{p,q}^a z^p w^q$$

where $\kappa_n^a = \kappa_n(a,\ldots,a)$ and $\kappa_{p,q}^a = \kappa_{p,q}(a,\ldots,a)$.
Moment series,

$$G(z) = \frac{1}{z}\sum_{n\geq 0}\tau(a^n)z^{-n}, \qquad G(z,w) = \frac{1}{zw}\sum_{n,m\geq 1}\tau(a^p,a^q)z^{-p}w^{-q}.$$

Then we have the relations

$$\frac{1}{G(z)} + R(G(z)) = z$$

and

$$G(z,w) = G'(z)G'(w)R(G(z),G(w)) + \frac{\partial^2}{\partial z\partial w}\log\left(\frac{G(z)-G(w)}{z-w}\right).$$

Second order freeness

Definition

The random variables $a_1, ...a_n$ are second order free if

- They are free.
- The second order **mixed** cumulants vanish.

Second order freeness

Definition

The random variables $a_1, ...a_n$ are second order free if

- They are free.
- The second order **mixed** cumulants vanish.

- Matrices are asymptoticallly second order free if the first and second order mixed cumulants vanish in the limit.
- Examples. Complex Gaussian matrices, and Wishart, (UAU^*, B) with A and B deterministic.
  Non examples: General Wigner matrices.

Theorem (Mingo Speicher Tan 08)

Suppose $n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+s}$ *are positive integers,* $p = n_1 + \cdots n_r, q = n_{r+1} + \cdots + n_{r+s}$, *and*

$$N = \{n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{r+s}\}$$

Given a second probability space $(\mathcal{A}, \tau, \tau_2)$ *and*

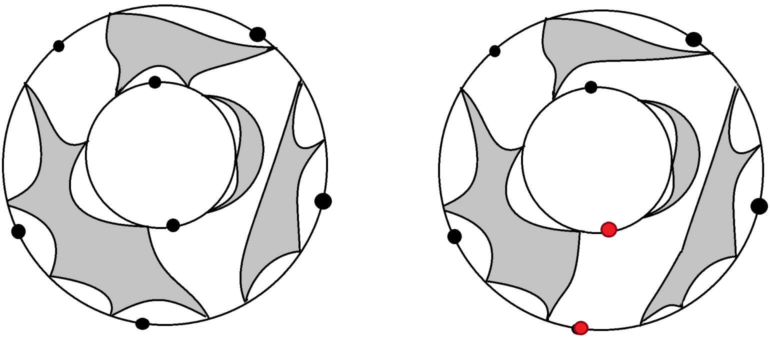
$$a_1, \ldots, a_{n_1}, a_{n_1+1}, \ldots, a_{n_1+n_2}, \ldots, a_{n_1+\cdots+n_{r+s}} \in \mathcal{A}$$

let $A_1 = a_1 \cdots a_{n_1}, A_2 = a_{n_1+1} \cdots a_{n_1+n_2}, \ldots, A_{r+s} = a_{n_1+\cdots+n_{r+s-1}+1} \cdots a_{n_1+\cdots+n_{r+s}}$. *Then*

$$\kappa_{r,s}(A_1, \ldots, A_r, A_{r+1}, \ldots, A_{r+s}) = \sum_{(V,\pi)} \kappa_{(V,\pi)}(a_1, \ldots, a_{p+q}), \quad (4)$$

where the summation is over those $(V, \pi) \in \mathcal{PS}_{NC}(p, q)$ *such that* $\pi^{-1}\gamma_{p,q}$ *separates the points of* N.

Separating the points



Second Order diagonal and Even elements

R-diagonal elements

Definition

Let (A, τ, τ_2) be a second order non-commutative probability space. An element $a \in (\mathcal{A}, \tau, \tau_2)$ is called **second order** R-**diagonal** if it is R-diagonal and the only non-vanishing second order cumulants are of the form

$$\kappa_{2p,2q}(a, a^*, \ldots, a, a^*) = \kappa_{2p,2q}(a^*, a, \ldots, a^*, a).$$

Examples:

- Second order Haar Unitary: Limiting distribution of Haar distributed on U(n). (not obvious)
- Circular operator c: Limiting distribution of $G_1 + iG_2$, GUE matrices.

Definition

Let (A, τ, τ_2) be a second order non-commutative probability space. An element $x \in (\mathcal{A}, \tau, \tau_2)$ is called **second order even** (or even for short) if $x = x^*$ and x is such that odd moments vanish, i.e. $\tau_2(x^p, x^q) = 0$ unless p and q are even and $\tau(x^{2n+1}) = 0$ for all $n \geq 0$.

Example:

- Semicircular operator c: Limiting distribution of G a GUE matrices.

Theorem

Let a be second order R-diagonal in $(\mathcal{A}, \tau, \tau_2)$ *and consider the off-diagonal matrix,*

$$A := \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$$

as an element in $(M_2(\mathcal{A}), tr_2 \otimes \tau, \tilde{\tau}_2)$*. Then*

1. *A is second order even element.*
2. *A has the same determining sequence as a:* $\beta_{p,q}^{(A)} = \beta_{p,q}^{(a)}$ *for all* $p, q \geq 1$ *and* $\beta_n^{(A)} = \beta_n^{(a)}$ *for all n.*
3. *The second order cumulants of a and the second order cumulants of A, are related by*

$$\kappa_{p,q}^{(a)} = \kappa_{p,q}^{(A)} + \sum_{\pi \in S_{NC}^{all,+}(m,n)} \kappa_{\pi}^{(A)}.$$

multiplication with R-diagonal

Theorem

Let $\{a, a^*\}$ and $\{b, b^*\}$ be second order free and suppose that a is second order R-diagonal. Then ab is second order R-diagonal.

Examples:

- $c_1 c_2 \cdots c_n$, c_i second order Circular Operator.
- ua, u second order Haar, a limit of A_n deterministic.

Definition

Let (A, τ, τ_2) be a second order non-commutative probability space. Let a be second order R-diagonal. Define $\beta_n^{(a)} := \kappa_{2n}(a, a^*, \ldots, a, a^*)$ and

$$\beta_{p,q}^{(a)} := \kappa_{2p,2q}(a, a^*, \ldots, a, a^*). \tag{5}$$

Theorem

Let a be a second order R-diagonal with determining sequences $(\beta_n^{(a)})_{n\geq 1}$ and $(\beta_{p,q}^{(a)})_{p,q\geq 1}$ then we have

$$\kappa_{p,q}(aa^*, \ldots, aa^*) = \sum_{(\mathcal{V},\pi)\in\mathcal{PS}_{NC}(p,q)} \beta_{(\mathcal{V},\pi)}^{(a)}. \tag{6}$$

$$\kappa_{p,q}(aa^*, \dots, aa^*) = \sum_{(\mathcal{V},\pi) \in PS_{NC}(p,q)} \beta^{(a)}_{(\mathcal{V},\pi)}.$$

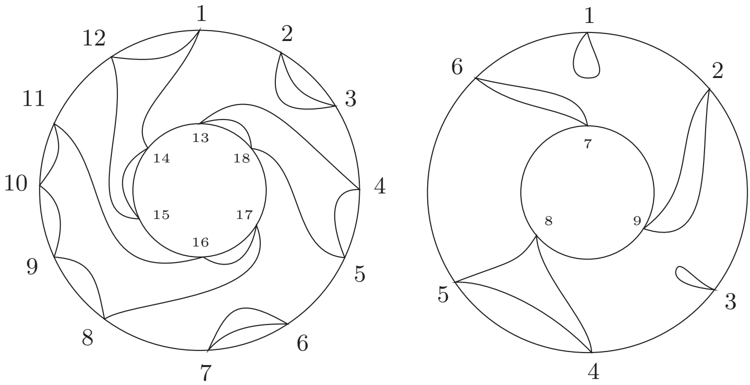


Figure: On the left we see the permutation $\pi = (1, 14, 15, 12)(2, 3)(4, 5, 18, 13)(6, 7)(8, 9, 10, 11, 16, 17)$. π is in $S^-_{NC}(12, 6)$ and $\gamma_{12,6}\pi^{-1}$ separates the points of $O = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$. On the right is $\check{\pi} = (1)(2, 9)(3)(4, 5, 8)(6, 7)$. to the even numbers.

Definition

Let (A, τ, τ_2) be a second order non-commutative probability space. Let x be a second order even element. Define $\beta_n^{(x)} := \kappa_{2n}^{(x)} := \kappa_{2n}(x, \ldots, x)$ and letting $\kappa_{p,q}^{(x)} = \kappa_{p,q}(x, \ldots, x)$, we set

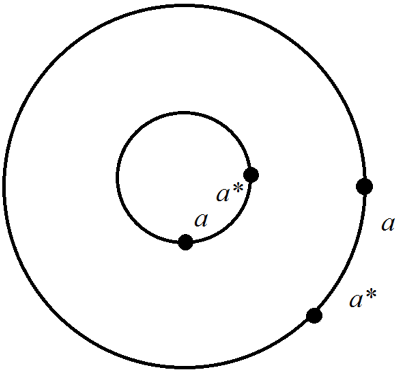
$$\beta_{p,q}^{(x)} := \kappa_{2p,2q}^{(x)} + \sum_{\pi \in S_{NC}^{all,+}(2p,2q)} \kappa_{\pi}^{(x)} \tag{7}$$

Theorem

Let x be an even element with determining sequences $(\beta_n^{(x)})_{n\geq 1}$ and $(\beta_{p,q}^{(x)})_{p,q\geq 1}$ then the second order cumulants of x^2 are given by

$$\kappa_{p,q}(x^2, \ldots, x^2) = \sum_{(\mathcal{V},\pi) \in PS_{NC}(p,q)} \beta_{(\mathcal{V},\pi)}^{(x)}. \tag{8}$$

Main difference in using Mingo-Speicher-Tan formula:



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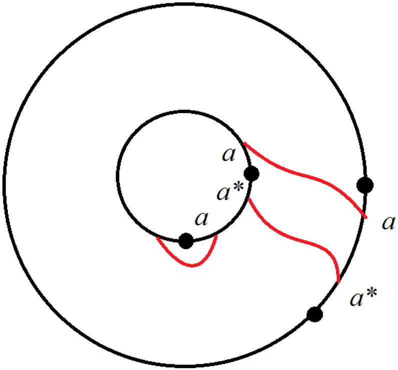

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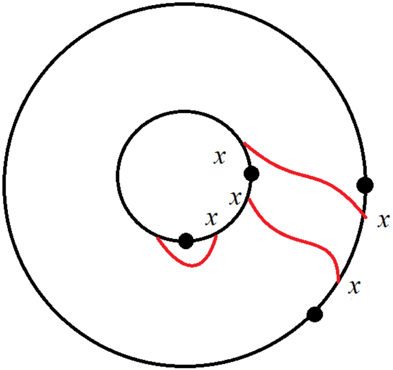

Theorem

Let x be either second order even or second order R-diagonal element with determining sequences $(\beta_n)_{n\geq 1}$ and $(\beta_{p,q})_{p,q\geq 1}$, define the formal power series

$$B(z)=\frac{1}{z}\sum_{n\geq 1}\beta_n z^n, \qquad B(z,w)=\frac{1}{zw}\sum_{p,q\geq 1}\beta_{p,q}z^p w^q,$$

$$C(z)=\frac{1}{z}+\frac{1}{z}\sum_{n\geq 1}\kappa_n^{xx^*}z^{-n}, \qquad C(z,w)=\frac{1}{zw}\sum_{n,m\geq 1}\kappa_{p,q}^{xx^*}z^{-p}w^{-q}.$$

Then

$$\frac{1}{C(z)}+B(C(z))=z$$

and

$$C(z,w)=C'(z)C'(w)B(C(z),C(w))+\frac{\partial^2}{\partial z\partial w}\log\left(\frac{C(z)-C(w)}{z-w}\right).$$

Examples

- s is called a *second order semi-circular operator* if its first order cumulants satisfy $\kappa_n(s, s, ..., s) = 0$ for all $n \neq 2$ and $\kappa_2(s, s) = 1$, and for all p and q the second order cumulants $\kappa_{p,q}$ are 0. This operator appears as the limit of GUE random matrices as the size tends to infinity.
- Consider s_1 and s_2 second order free semicircular operators. We call $c = \frac{s_1 + is_2}{\sqrt{2}}$ a (second order) circular operator. The operator c is a second order R-diagonal. Indeed, since s_1 and s_2 are second order free, their mixed free cumulants vanish, also the second order cumulant of s_1 and s_2 vanish thus by linearity the same is true for c. That is, $\kappa_{p,q}(c^{(\epsilon_1)}, ..., c^{(\epsilon_{p+q})}) = 0$ for all $\epsilon_1, ..., \epsilon_{p+q} \in \{\pm 1\}$.

Square of semicircular

The determining sequence of a semicircle operator is given by $\beta_1 = 1$ and $\beta_n = 0$ for $n > 0$. The second order determining sequence is given by $\beta_{k,k} = k$ and $\beta_{p,q} = 0$ if $p \neq q$.Thus second order cumulants of s^2.

$$
\begin{aligned}
&\kappa_{p,q}(s^2,\ldots,s^2) \\
&= \sum_{\pi \in S_{NC}(p,q)} \beta_\pi + \sum_{(\mathcal{V},\pi)\in \mathcal{PS}_{NC}(p,q)'} \beta_{(\mathcal{V},\pi)} = \sum_{\pi \in \mathcal{PS}_{NC}(p,q)'} \beta_{(\mathcal{V},\pi)} \\
&= \sum_{k>0} \binom{p}{k}\binom{q}{k} \beta_{k,k} = \sum_{k>0} k \binom{p}{k}\binom{q}{k} = p\binom{p+q-1}{p}.
\end{aligned}
$$

Square of circular

- Recall that cc^* and s^2 both have a free Poisson distribution with respect to τ (i.e. $\tau((cc^*)^n) = \tau((s^2)^n) = \frac{1}{n+1}\binom{2n}{n}$). So one might expect that cc^* and s^2 have the same distribution in the second order level. We see that this is not the case.
- Indeed, the determining sequences of c are given by $\beta_1 = 1$ and $\beta_n = 0$ for $n > 0$ and $\beta_{p,q} = 0$, for all p and q. It is readily seen that the second order cumulants of cc^* are all zero. i.e. $\kappa_{p,q}(cc^*, cc^*, ..., cc^*) = 0$.
- Since the first order cumulants of cc^* are all 1. Then the (p, q)-fluctuation moments of cc^* count the number of elements in $S_{NC}(p, q)$.

Product of k second order free operators

Theorem (Second order Moments and Cumulants of Products of Free Variables)

Let $a_1, \ldots, a_k$ be operators which are second order free and such that $\kappa_{p,q}^{(a_i)} = 0$ for all p and q. Denote by $a := a_1 a_2 \cdots a_k$.

$$\tau(a^p, a^q) = \sum_{\pi \in S_{NC}^{k\text{-alt}}(p,q)} \kappa_{Kr(\pi)}(a, \ldots, a). \qquad (9)$$

Furthermore,

$$\kappa_{p,q}^{(a)} = \sum_{\pi \in S_{NC}^{k\text{-alt-eq}}(p,q)} \kappa_{Kr(\pi)}(a, \ldots, a). \qquad (10)$$

Let $h = c_1 c_2 \cdots c_k$. Equations (9) and (10) of Theorem 26 are very useful. Indeed, a direct application of them gives a combinatorial description of the fluctuation moments and cumulants:

$$\tau_2(h^p, h^q) = |SNC^k_{alt}(kp, kq)| \text{ and } \kappa_{p,q}(h, ..., h) = |SNC_{k\ alt}(kp, kq)|,$$

a precise formula for this quantitiesor a generating function is not known for $k > 2$.

The c be a 2nd order circular operator which is second order $*$-free from a. We are interested in the second order cumulants of cac^*. We can show that as in the first order level

$$\kappa_{p,q}(cac^*, ..., cac^*) = \sum_{(\mathcal{V},\pi)\in\mathcal{PS}_{NC}(p,q)} \kappa_{(\mathcal{V},\pi)}(a, ..., a) = \tau_2(a^p, a^q).$$

A particular important example is the case when the fluctuation moments of a are 0. In this case $G_a(z, w) = 0$ and the the formula above is reduced to

$$G_{cac^*}(z, w) = \frac{\partial^2}{\partial z \partial w} \log\left(\frac{G_{cac^*}(z) - G_{cac^*}(w)}{z - w}\right).$$

From the random matrix perspective this corresponds to the case when a is a limit of deterministic matrices and cac^* then corresponds to WAW^* where W is a Ginibre matrix and A is deterministic.

Counter-example

Two desirable properties for R-diagonal are not true in the second order level.

- If r is 2nd order R diagonal r^n is not necesarily 2nd order R-diagonal.
- If u is 2nd order free from a. ubu^* may not be 2nd order free from a.

Thanks!