Multiplicative and semi-multiplicative functions on non-crossing partitions, and relations to cumulants

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Summary:

We are presenting arXiv:2106.16072. This is joint work with

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Leon Witzman (Waterloo undergraduate in 2020-2021).

Goal of talk and underlying paper: look at the old topic around "multiplicative functions in an incidence algebra". Look at some convolution groups appearing in that framework, which can be used to study the newer topic of relations between different brands of cumulants.
Part 2 of the talk: Hopf algebra facet of this development – groups from Part 1 appearing now as group of characters for some natural Hopf algebra constructions.

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Sequences of multilinear functionals.

A noncommutative probability space $(\mathcal{A}, \varphi)$ has a sequence of moment functionals $(\varphi_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$, with
$\varphi_n(x_1, \ldots, x_n) := \varphi(x_1 \cdots x_n), \ \ \forall\, n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{A}$.

$(\mathcal{A}, \varphi)$ also has free cumulant functionals $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$.
The formulas connecting φ_n's to κ_n's live in the framework of the non-crossing partitions $NC(n)$.

Other brands of cumulant functionals of $(\mathcal{A}, \varphi)$ living in the world of $NC(n)$:

- Boolean cumulants $(\beta_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$.
- Monotone cumulants $(\rho_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$.
- For a given a parameter $t \in \mathbb{R}$, one has "t-Boolean" cumulants $(\beta_n^{(t)} : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$.
  (Have $\beta_n^{(0)} = \beta_n$ and $\beta_n^{(1)} = \kappa_n$.)

Not living in the world of $NC(n)$: classical cumulants,
q-free cumulants.

Transition between two sequences of multilinear functionals.
Have transition formulas: moments↔cumulants. Also have transition formulas between different brands of cumulants (cf. [AHLV2015]). Look at general pattern of how this is done.

• First, standard move: a sequence $(\psi_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$, extends to a larger family $(\psi_\pi : \mathcal{A}^n \to \mathbb{C})_{n\geq 1,\, \pi \in NC(n)}$.
[Need here some review of $NC(n)$-terminology – next slide.]

• Suppose we are given two sequences of functionals, $(\psi_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ and $(\theta_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$. Transition from the first sequence to the second is made by equations of the form

$$(1) \qquad \theta_n = \psi_n + \sum_{\pi \in NC(n),\, |\pi|\geq 2} \alpha_\pi \psi_\pi, \quad \forall\, n \in \mathbb{N}$$

(equality of n-linear functionals on $\mathcal{A}$, where the α_π's are some complex coefficients).

Point to follow: Eqn.(1) records a certain *group action*! The group will be denoted as $\widetilde{\mathcal{G}}$ (does not depend on $\mathcal{A}$).

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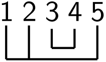
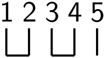
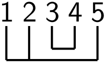
Towards $\widetilde{\mathcal{G}}$: $NC(n)$'s, and unitized functions on $NC^{(2)}$.

• Denote $NC(n)$:= set of all *non-crossing partitions* of $\{1, \ldots, n\}$.
E.g. have $\sigma =$ 

$\in NC(5)$. The sets $\{1, 2, 5\}$ and $\{3, 4\}$ are called *blocks* of this σ, and $\{3, 4\}$ is *nested* inside $\{1, 2, 5\}$. $\{1, 2, 5\}$ is an *outer* block, while $\{3, 4\}$ is an *inner* block.

• $NC(n)$ is partially ordered by *reverse refinement*: "$\pi \leq \sigma$" means that every block of π is contained in a block of σ. E.g. have $\pi \leq \sigma$ in $NC(5)$ for $\pi =$ 

, $\sigma =$ 

. Here π is an example of *interval partition* (all blocks are outer – no nestings).

• Will denote $NC^{(2)} := \sqcup_{n=1}^{\infty} \{(\pi, \sigma) \mid \pi, \sigma \in NC(n), \pi \leq \sigma\}$,
and $\mathcal{F} := \left\{ f : NC^{(2)} \to \mathbb{C} \mid f(\pi, \pi) = 1 \text{ for all } \pi \in \sqcup_{n=1}^{\infty} NC(n) \right\}$.

Towards $\widetilde{\mathcal{G}}$: the groups $\mathcal{F}$ (too large) and $\mathcal{G}$ (too small).

- Denote $NC^{(2)} := \sqcup_{n=1}^{\infty}\{(\pi,\sigma) \mid \pi,\sigma \in NC(n), \pi \leq \sigma\}$, and

  $\mathcal{F} := \{ f : NC^{(2)} \to \mathbb{C} \mid f(\pi,\pi) = 1 \text{ for all } \pi \in \sqcup_{n=1}^{\infty} NC(n) \}$.

- For $f_1, f_2 \in \mathcal{F}$ define convolution $f_1 * f_2 \in \mathcal{F}$: for $n \geq 1$ and $\pi \leq \sigma$ in $NC(n)$ put $(f_1 * f_2)(\pi,\sigma) := \sum_{\substack{\rho \in NC(n), \\ \pi \leq \rho \leq \sigma}} f_1(\pi,\rho)\, f_2(\rho,\sigma)$.

Fact. $(\mathcal{F}, *)$ is a group. (Think: upper triangular matrices...)

Inside $\mathcal{F}$ identify certain "multiplicative" functions (they will give a smaller group, $\mathcal{G}$).

Fact. Let $\pi < \sigma$ (i.e. $\pi \leq \sigma$ and $\pi \neq \sigma$) be in $NC(n)$. Consider the interval $[\pi,\sigma] = \{\rho \in NC(n) \mid \pi \leq \rho \leq \sigma\}$. There exist $p_2, \ldots, p_n \in \mathbb{N} \cup \{0\}$, uniquely determined, such that

$$[\pi,\sigma] \approx NC(2)^{p_2} \times \cdots \times NC(n)^{p_n} \qquad (2)$$

(poset isomorphism, with natural partial orders based on reverse refinement on both sides).

Towards $\widetilde{\mathcal{G}}$: the groups $\mathcal{F}$ (too large) and $\mathcal{G}$ (too small).

Recap: • $\mathcal{F} = \{ f : NC^{(2)} \to \mathbb{C} \mid f(\pi, \pi) = 1 \text{ for all } \pi \}$.
• For $\pi < \sigma$ in $NC(n)$ have canonical isomorphism

$$(2) \qquad [\pi, \sigma] \approx NC(2)^{p_2} \times \cdots \times NC(n)^{p_n},$$

with uniquely determined $p_2, \ldots, p_n \in \mathbb{N} \cup \{0\}$.

Definition. 1^o Suppose we are given a sequence $(\alpha_n)_{n=1}^{\infty}$ with α_n's in $\mathbb{C}$, where $\alpha_1 = 1$. Define $g : NC^{(2)} \to \mathbb{C}$ as follows: for $\pi < \sigma$ in $NC(n)$ put $g(\pi, \sigma) = \alpha_2^{p_2} \cdots \alpha_n^{p_n}$, with $p_2, \ldots, p_n$ as in (2). Define moreover $g(\pi, \pi) = 1$ for all $\pi \in \sqcup_{n=1}^{\infty} NC(n)$. Have $g \in \mathcal{F}$; it is said to be *multiplicative*.

2^o Let $\mathcal{G} :=$ (set of all functions g defined as in 1^o).

Fact. $\mathcal{G}$ is a subgroup of $(\mathcal{F}, *)$.

[General idea of such $\mathcal{F}$ and $\mathcal{G}$, in a family of lattices: Rota, 1960's. Case of $NC(n)$'s and use in free probability: Speicher, 1990's.]

Semi-multiplicative functions and the group $\widetilde{\mathcal{G}}$.

The poset isomorphism $[\pi, \sigma] \approx NC(2)^{p_2} \times \cdots \times NC(n)^{p_n}$ is easy to describe explicitly, in two steps.

Step 1. Factor $[\pi, \sigma]$ into "intervals-to-the-top":

$$(3) \qquad [\pi, \sigma] \approx [\rho_1, 1_{m_1}] \times \cdots \times [\rho_k, 1_{m_k}]$$

with $m_1, \ldots, m_k \geq 1$ and $\rho_1 \in NC(m_1), \ldots, \rho_k \in NC(m_k)$, where $1_m \in NC(m)$ is notation for partition with 1 block.
[Notes: – For $\rho \in NC(m)$ have $[\rho, 1_m] = \{\theta \in NC(m) \mid \theta \geq \rho\}$.
– The $m_1, \ldots, m_k$ in (3) are the sizes of the blocks of σ.]

Step 2. Factor: $[\rho, 1_m] \approx NC(2)^{q_2} \times \cdots \times NC(m)^{q_m}$. Done with help of a special anti-automorphism of $NC(m)$, the Kreweras complementation map.

Definition. A function $g \in \mathcal{F}$ is said to be *semi-multiplicative* when it respects (3): $g(\pi, \sigma) = g(\rho_1, 1_{m_1}) \cdots g(\rho_k, 1_{m_k})$.
Denote $\widetilde{\mathcal{G}} := \{g \in \mathcal{F} \mid g \text{ is semi-multiplicative}\}$.

The group $\widetilde{\mathcal{G}}$ and its action on $\mathfrak{M}_{\mathcal{A}}$.

Recap: • $g \in \mathcal{F}$ is *semi-multiplicative* when it complies with Step 1 in factorization (but no requirement concerning Step 2).
• $\widetilde{\mathcal{G}} := \{g \in \mathcal{F} \mid g \text{ is semi-multiplicative}\}$. Have $\mathcal{G} \subseteq \widetilde{\mathcal{G}} \subseteq \mathcal{F}$.

Proposition. *$\widetilde{\mathcal{G}}$ is a subgroup of $(\mathcal{F}, *)$.*

Next: $\widetilde{\mathcal{G}}$ has a natural action, on the right, on the sequences of multilinear functionals described in Section 1 of the talk.

Notation. For $\mathcal{A}$ a vector space over $\mathbb{C}$ denote
$\mathfrak{M}_{\mathcal{A}} := \{\underline{\psi} = (\psi_n)_{n=1}^{\infty} \mid \psi_n : \mathcal{A}^n \to \mathbb{C} \text{ multilinear, for all } n \geq 1\}$.

For $\underline{\psi} \in \mathfrak{M}_{\mathcal{A}}$ and $g \in \widetilde{\mathcal{G}}$ define $\underline{\psi} \cdot g := \underline{\theta} = (\theta_n)_{n=1}^{\infty}$, where

$$(4) \qquad \theta_n = \sum_{\pi \in NC(n)} g(\pi, 1_n)\, \psi_\pi, \quad n \geq 1.$$

[Notes: – In (4) we have an equality of n-linear functionals on $\mathcal{A}$.
– For $\underline{\psi}$ we use the extended family of multilinear functionals, as in Section 1, $(\psi_\pi : \mathcal{A}^n \to \mathbb{C})_{n \geq 1,\, \pi \in NC(n)}$.]

The group $\widetilde{\mathcal{G}}$ and its action on $\mathfrak{M}_{\mathcal{A}}$.

Clarify Eqn.(4), which says: $\theta_n = \sum_{\pi \in NC(n)} g(\pi, 1_n)\, \psi_\pi, \quad n \geq 1$.

- What is $\psi_\pi : \mathcal{A}^n \to \mathbb{C}$ for a partition $\pi \in NC(n)$:

$$\psi_\pi(x_1, \ldots, x_n) := \prod_{V \text{ block of } \pi} \psi_{|V|}\big((x_1, \ldots, x_n) \mid V \big).$$

E.g. for $\pi =$  we get $\psi_\pi(x_1, \ldots, x_5) = \psi_3(x_1, x_2, x_5) \cdot \psi_2(x_3, x_4)$.

- A concrete instance of (4), with $n = 3$: have

$$\theta_3(x_1, x_2, x_3) = \psi_3(x_1, x_2, x_3) + g(0_3, 1_3)\, \psi_1(x_1)\psi_1(x_2)\psi_1(x_3)$$
$$+ g(\pi_1, 1_3)\, \psi_1(x_1)\psi_2(x_2, x_3) + \cdots \text{(2 more terms)},$$

where $0_3 := \big\{\{1\}, \{2\}, \{3\}\big\}$, $\pi_1 := \big\{\{1\}, \{2, 3\}\big\}$ (and analogous π_2, π_3 for the 2 more terms).

Proposition. *Equation (4) describes a group action:*

$$(\underline{\psi} \cdot g_1) \cdot g_2 = \underline{\psi} \cdot (g_1 * g_2), \text{ for } \underline{\psi} \in \mathfrak{M}_{\mathcal{A}} \text{ and } g_1, g_2 \in \widetilde{\mathcal{G}}.$$

Basic examples of how g's in $\widetilde{\mathcal{G}}$ encode transitions.

Go back to framework of Section 1: $(\mathcal{A}, \varphi)$ noncommutative prob. space; look at the sequence $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ of moment functionals, where $\varphi_n : \mathcal{A}^n \to \mathbb{C}$ is $\varphi_n(x_1, \ldots, x_n) = \varphi(x_1 \cdots x_n)$.

Example 1. $(\mathcal{A}, \varphi)$ also has a sequence of free cumulant functionals $\underline{\kappa} = (\kappa_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$, defined via the requirement that

$$\varphi_n = \sum_{\pi \in NC(n)} \kappa_\pi, \quad \forall\, n \geq 1.$$

This requirement can be re-phrased as $\underline{\varphi} = \underline{\kappa} \cdot g_{\mathrm{fc-m}}$, where $g_{\mathrm{fc-m}} \in \widetilde{\mathcal{G}}$ is defined by putting $g_{\mathrm{fc-m}}(\pi, \sigma) = 1$, for all $n \geq 1$ and $\pi \leq \sigma$ in $NC(n)$.

Basic examples of how g's in $\widetilde{\mathcal{G}}$ encode transitions.

Continue with $(\mathcal{A}, \varphi)$ and $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ as above.

Example 2. $(\mathcal{A}, \varphi)$ also has a sequence of Boolean cumulant functionals $\underline{\beta} = (\beta_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$, defined via the requirement that

$$\varphi_n = \sum_{\pi \in \mathrm{Int}(n)} \beta_\pi, \quad \forall\, n \geq 1,$$

where $\mathrm{Int}(n) := \{\pi \in NC(n) \mid \pi \text{ is an interval partition}\}$. This requirement can be re-phrased as $\underline{\varphi} = \underline{\beta} \cdot g_{\mathrm{bc-m}}$, where $g_{\mathrm{bc-m}}$ is defined as follows: it is the unique function in $\widetilde{\mathcal{G}}$ which has, for every $n \geq 1$ and $\pi \in NC(n)$,

$$g_{\mathrm{bc-m}}(\pi, 1_n) = \begin{cases} 1, & \text{if } \pi \in \mathrm{Int}(n), \\ 0, & \text{otherwise.} \end{cases}$$

Basic examples of how g's in $\widetilde{\mathcal{G}}$ encode transitions.

Example 3. Continue with $(\mathcal{A}, \varphi)$ as above, and with the two sequences of cumulant functionals $\underline{\kappa} = (\kappa_n)_{n=1}^{\infty}$ and $\underline{\beta} = (\beta_n)_{n=1}^{\infty}$. Have a nice direct transition from $\underline{\kappa}$ to $\underline{\beta}$, found by Lehner [L2002]:

$$\beta_n = \sum_{\pi \in NC_{\mathrm{irr}}(n)} \kappa_\pi, \quad \forall\, n \geq 1,$$

where $NC_{\mathrm{irr}}(n) := \{\pi \in NC(n) \mid 1 \text{ and } n \text{ in same block of } \pi\}$. This can be re-phrased as $\underline{\beta} = \underline{\kappa} \cdot g_{\mathrm{fc-bc}}$, where $g_{\mathrm{fc-bc}}$ is the unique function in $\widetilde{\mathcal{G}}$ which has, for every $n \geq 1$ and $\pi \in NC(n)$,

$$g_{\mathrm{fc-bc}}(\pi, 1_n) = \begin{cases} 1, & \text{if } \pi \in NC_{\mathrm{irr}}(n), \\ 0, & \text{otherwise.} \end{cases}$$

Note: $g_{\mathrm{fc-m}} = g_{\mathrm{fc-bc}} * g_{\mathrm{bc-m}}$ (relation in $\widetilde{\mathcal{G}}$ – can also be checked by direct combinatorics).

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Remark. A function $g \in \widetilde{\mathcal{G}}$ is completely determined when we know the values $g(\pi, 1_n)$ for all $n \geq 1$ and $\pi \in NC(n)$. From there, all $g(\pi, \sigma)$ with $\pi \leq \sigma$ follow via semi-multiplicativity. Want to look at g's that are completely determined when we know (even less) the values $g(\pi, 1_n)$ with π in a smaller set $NC_{\mathrm{irr}}(n)$.

Notation *(Concatenation and irreducibility in $NC(n)$).*
1^o For $\pi_1 \in NC(n_1)$, $\pi_2 \in NC(n_2)$ can consider the concatenation $\pi_1 \diamond \pi_2 \in NC(n_1 + n_2)$: place π_1 on the points $1, \ldots, n_1$ and π_2 on $n_1 + 1, \ldots, n_1 + n_2$.

2^o Say that $\pi \in NC(n)$ is *irreducible* when one cannot write $\pi = \pi_1 \diamond \pi_2$ with $\pi_1 \in NC(n_1)$ and $\pi_2 \in NC(n_2)$ for some $n_1, n_2 \geq 1$. Denote $NC_{\mathrm{irr}}(n) := \{\pi \in NC(n) \mid \pi \text{ is irreducible}\}$.

[Notes: – For $\pi \in NC(n)$ have that

$$\big(\pi \text{ irreducible}\big) \Leftrightarrow \big(1 \text{ and } n \text{ are in same block of } \pi\big).$$

– Every $\pi \in \sqcup_{n=1}^{\infty} NC(n)$ can be written as a concatenation of irreducible partitions. How: follow the outer blocks of π...]

The subsets $\widetilde{\mathcal{G}}_{\mathrm{c-m}}$ and $\widetilde{\mathcal{G}}_{\mathrm{c-c}}$ of $\widetilde{\mathcal{G}}$.

Definition. 1^o $g \in \widetilde{\mathcal{G}}$ said to be of *cumulant-to-moment* type when it satisfies: $g(\pi_1 \diamond \pi_2, 1_{n_1+n_2}) = g(\pi_1, 1_{n_1}) \cdot g(\pi_2, 1_{n_2})$, for all $n_1, n_2 \geq 1$ and $\pi_1 \in NC(n_1)$, $\pi_2 \in NC(n_2)$. We denote

$$\widetilde{\mathcal{G}}_{\mathrm{c-m}} := \{g \in \widetilde{\mathcal{G}} \mid g \text{ is of cumulant-to-moment type}\}.$$

2^o $g \in \widetilde{\mathcal{G}}$ said to be of *cumulant-to-cumulant* type when it satisfies: $g(\pi_1 \diamond \pi_2, 1_{n_1+n_2}) = 0$, for all $n_1, n_2 \geq 1$ and $\pi_1 \in NC(n_1)$, $\pi_2 \in NC(n_2)$. (Equivalently: $g(\pi, 1_n) = 0$ whenever $\pi \in NC(n)$ is not irreducible.) We denote

$$\widetilde{\mathcal{G}}_{\mathrm{c-c}} := \{g \in \widetilde{\mathcal{G}} \mid g \text{ is of cumulant-to-cumulant type}\}.$$

Remark. A function $g \in \widetilde{\mathcal{G}}_{\mathrm{c-c}}$ is completely determined when we know its values $g(\pi, 1_n)$ with $n \geq 1$ and $\pi \in NC_{\mathrm{irr}}(n)$. Same is the case for a $g \in \widetilde{\mathcal{G}}_{\mathrm{c-m}}$.

The subgroup $\widetilde{\mathcal{G}}_{\mathbf{c-c}}$ of $\widetilde{\mathcal{G}}$, and its coset $\widetilde{\mathcal{G}}_{\mathbf{c-m}}$.
Rationale for notation: salient examples of cumulants living in the world of $NC(n)$'s fit the $\widetilde{\mathcal{G}}_{\mathrm{c-m}}$ and $\widetilde{\mathcal{G}}_{\mathrm{c-c}}$ terminology (cf. Section 7 of the paper). In particular have $g_{\mathrm{fc-m}}, g_{\mathrm{bc-m}} \in \widetilde{\mathcal{G}}_{\mathrm{c-m}}$ and $g_{\mathrm{fc-bc}} \in \widetilde{\mathcal{G}}_{\mathrm{c-c}}$.

Proposition. 1^o *$\widetilde{\mathcal{G}}_{\mathrm{c-c}}$ is a subgroup of $(\widetilde{\mathcal{G}}, *)$.*
2^o *$\widetilde{\mathcal{G}}_{\mathrm{c-m}}$ is a right coset of the subgroup $\widetilde{\mathcal{G}}_{\mathrm{c-c}}$. That is: we have*

$$\widetilde{\mathcal{G}}_{\mathrm{c-m}} = \widetilde{\mathcal{G}}_{\mathrm{c-c}} * h = \{ g * h \mid g \in \widetilde{\mathcal{G}}_{\mathrm{c-c}} \},$$

for any $h \in \widetilde{\mathcal{G}}_{\mathrm{c-m}}$ that we choose to fix.

Remark. Easiest choice of $h \in \widetilde{\mathcal{G}}_{\mathrm{c-m}}$: take $h(\pi, \sigma) = 1$ for all $(\pi, \sigma) \in NC^{(2)}$. That is, take $h = g_{\mathrm{fc-m}}$, giving the transition from free cumulants to moments.
In some ways it turns out to be more convenient to use $h = g_{\mathrm{bc-m}}$, the function which gives transition from Boolean cumulants to moments. Detailed discussion in paper (cf. Section 8.3 there.)

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t-Boolean cumulants.

Use framework of Section 1: $(\mathcal{A}, \varphi)$ noncommutative probability space, and look at the sequence $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ of moment functionals, where $\varphi_n : \mathcal{A}^n \to \mathbb{C}$ is $\varphi_n(x_1, \ldots, x_n) = \varphi(x_1 \cdots x_n)$.

Definition. Let $t \in \mathbb{R}$ be a parameter. We use the name *t-Boolean cumulant functionals* for the sequence of multilinear functionals $\underline{\beta}^{(t)} = (\beta_n^{(t)})_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ defined via the requirement that

$$(5) \qquad \varphi_n = \sum_{\pi \in NC(n)} t^{\mathrm{inner}(\pi)} \beta_\pi^{(t)}, \quad \forall\, n \geq 1,$$

where $\mathrm{inner}(\pi) :=$ (number of inner blocks of $\pi \in NC(n)$).

Remark. For $t = 1$, Eqn.(5) defines the free cumulants of $(\mathcal{A}, \varphi)$; hence $\underline{\beta}^{(1)} = \underline{\kappa}$. For $t = 0$ one gets that $\underline{\beta}^{(0)} = \underline{\beta}$, because in this case the RHS of Eqn.(5) reduces to the sum over $\mathrm{Int}(n)$ used to define Boolean cumulants.

This interpolation between free and Boolean cumulants arises in work of Bożejko and Wysoczanski [BW2001].

Enter the functions $\boldsymbol{g}^{(\boldsymbol{t})}_{\mathbf{bc-m}} \in \widetilde{\boldsymbol{\mathcal{G}}}_{\mathbf{c-m}}$.

Notation. For every $t \in \mathbb{R}$, let $g^{(t)}_{\mathrm{bc-m}} \in \widetilde{\mathcal{G}}$ be defined via the requirement that $g^{(t)}_{\mathrm{bc-m}}(\pi, 1_n) := t^{\mathrm{inner}(\pi)}$, for all $n \geq 1$ and $\pi \in NC(n)$.

Remark. 1^o $g^{(1)}_{\mathrm{bc-m}} = g_{\mathrm{fc-m}}$ and $g^{(0)}_{\mathrm{bc-m}} = g_{\mathrm{bc-m}}$ from Examples 1 and 2 at the end of Section 2 of the talk.

2^o Obvious that we have $\mathrm{inner}(\pi_1 \diamond \pi_2) = \mathrm{inner}(\pi_1) + \mathrm{inner}(\pi_2)$, for all $\pi_1, \pi_2 \in \sqcup_{n=1}^{\infty} NC(n)$. This implies that $g^{(t)}_{\mathrm{bc-m}}$ is a function of cumulant-to-moment type, for every $t \in \mathbb{R}$.

Remark. Eqn.(5) of preceding slide, $\varphi_n = \sum_{\pi \in NC(n)} t^{\mathrm{inner}(\pi)} \beta^{(t)}_{\pi}$ for all $n \geq 1$, gets now to be concisely written in the form $\underline{\varphi} = \underline{\beta}^{(t)} \cdot g^{(t)}_{\mathrm{bc-m}}$. This is a common generalization of Examples 1 and 2 from Section 2 of the talk.

Transition formula from s-Boolean to t-Boolean?

General idea: use action of $\widetilde{\mathcal{G}}$ to combine moment-cumulant formulas for two different brands of cumulants, in order to get a direct connection between the cumulants themselves.

Here: pick $s, t \in \mathbb{R}$ and write: $\underline{\beta}^{(t)} \cdot g^{(t)}_{\mathrm{bc-m}} = \underline{\varphi} = \underline{\beta}^{(s)} \cdot g^{(s)}_{\mathrm{bc-m}}$,

hence:
$$\begin{aligned} \underline{\beta}^{(t)} &= \underline{\varphi} \cdot (g^{(t)}_{\mathrm{bc-m}})^{-1} \\ &= (\underline{\beta}^{(s)} \cdot g^{(s)}_{\mathrm{bc-m}}) \cdot (g^{(t)}_{\mathrm{bc-m}})^{-1} \\ &= \underline{\beta}^{(s)} \cdot \big(g^{(s)}_{\mathrm{bc-m}} * (g^{(t)}_{\mathrm{bc-m}})^{-1} \big). \end{aligned}$$

Conclusion: the transition from s-Boolean cumulants to t-Boolean cumulants is encoded by the function $g^{(s)}_{\mathrm{bc-m}} * (g^{(t)}_{\mathrm{bc-m}})^{-1} \in \widetilde{\mathcal{G}}_{\mathrm{c-c}}$.

But can we write explicitly what are the values of this function? Turns out that this can be nicely settled by using a suitable 1-parameter subgroup $(u_q)_{q \in \mathbb{R}}$ of $\widetilde{\mathcal{G}}_{\mathrm{c-c}}$.

A 1-parameter subgroup "generated by $g_{\mathrm{fc-bc}}$".

Notation. For every $q \in \mathbb{R}$, let u_q be the function in $\widetilde{\mathcal{G}}_{\mathrm{c-c}}$ defined via the requirement that for $n \geq 1$ and $\pi \in NC(n)$ we have

$$u_q(\pi, 1_n) = \begin{cases} q^{|\pi|-1}, & \text{if } \pi \text{ is irreducible} \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1. *$u_{q_1} * u_{q_2} = u_{q_1+q_2}$, for all $q_1, q_2 \in \mathbb{R}$.*

Remark. Immediate that for $q = 1$ we get $u_1 = g_{\mathrm{fc-bc}}$ from Example 3 at end of Section 2 (transition from free cumulants to Boolean cumulants). Proposition 1 thus implies that $u_q = g_{\mathrm{fc-bc}}^q$ for all $q \in \mathbb{Z}$. In a certain sense, $(u_q)_{q \in \mathbb{R}}$ is then a 1-parameter subgroup of $\widetilde{\mathcal{G}}_{\mathrm{c-c}}$ "generated by $g_{\mathrm{fc-bc}}$".

Proposition 2. *$u_q * g_{\mathrm{bc-m}}^{(t)} = g_{\mathrm{bc-m}}^{(q+t)}$, for all $q, t \in \mathbb{R}$.*

Transition formula from s-Boolean to t-Boolean.

Finalize discussion from two slides ago, about the cumulant functionals $\underline{\beta}^{(s)} = (\beta_n^{(s)})_{n=1}^{\infty}$ and $\underline{\beta}^{(t)} = (\beta_n^{(t)})_{n=1}^{\infty}$ on $(\mathcal{A}, \varphi)$.

Corollary. *Have* $\beta_n^{(t)} = \sum_{\substack{\pi \in NC(n), \\ \text{irreducible}}} (s-t)^{|\pi|-1} \beta_\pi^{(s)}$, $\forall\, n \geq 1$.

Proof. Conclusion from two slides ago was that: transition from $\underline{\beta}^{(s)}$ to $\underline{\beta}^{(t)}$ is encoded by the function $g_{\mathrm{bc-m}}^{(s)} * (g_{\mathrm{bc-m}}^{(t)})^{-1} \in \widetilde{\mathcal{G}}_{\mathrm{c-c}}$. But the above Proposition 2, with $q = s - t$, says that:

$$u_q * g_{\mathrm{bc-m}}^{(t)} = g_{\mathrm{bc-m}}^{(q+t)} = g_{\mathrm{bc-m}}^{(s)};$$

hence $g_{\mathrm{bc-m}}^{(s)} * (g_{\mathrm{bc-m}}^{(t)})^{-1} = u_{s-t}$. The transition from $\underline{\beta}^{(s)}$ to $\underline{\beta}^{(t)}$ takes the form $\underline{\beta}^{(t)} = \underline{\beta}^{(s)} \cdot u_{s-t}$. When written explicitly, this gives precisely the formula indicated in the corollary. □

Remark. In the special case when s and t are 0 and 1 (in some order) we get the known transition formulas between free and Boolean cumulants.

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The u_q's are in the normalizer of $\mathcal{G}$.

Consider the subgroup $\mathcal{G} \subseteq \widetilde{\mathcal{G}}$ which consists of multiplicative functions. On the other hand, consider the 1-parameter group $(u_q)_{q\in\mathbb{R}}$ (of the subgroup $\widetilde{\mathcal{G}}_{\mathrm{c-c}} \subseteq \widetilde{\mathcal{G}}$) which we had in Section 4 of the talk.

Theorem. $\Big(q \in \mathbb{R},\ f \in \mathcal{G}\Big) \ \Rightarrow\ u_q^{-1} * f * u_q \in \mathcal{G}.$

[Interesting underlying combinatorics, relying on a "relative" version of the Kreweras complementation map.]

The preceding theorem can be used in order to better understand the intriguing known fact that multiplication of freely independent random variables is nicely described in terms of Boolean cumulants (who aren't a priori meant to be related to free probability).

Formula for t-Boolean cumulants of $x \cdot y$.

Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, let $x, y \in \mathcal{A}$ be freely independent, and let $t \in \mathbb{R}$ be a parameter. Let $\underline{\beta}^{(t)} = (\beta_n^{(t)})_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ be the t-Boolean cumulant functionals of $(\mathcal{A}, \varphi)$.

Theorem. *For every $n \geq 1$, we have*

$$(6) \quad \beta_n^{(t)}(xy, \ldots, xy) = \sum_{\pi \in NC(n)} \beta_{\pi}^{(t)}(x, \ldots, x) \cdot \beta_{\mathrm{Kr}(\pi)}^{(t)}(y, \ldots, y),$$

where $\mathrm{Kr}(\pi)$ stands for the Kreweras complement of $\pi \in NC(n)$.

Point is: the formula describing the t-Boolean cumulants of $x \cdot y$ in terms of the separate t-Boolean cumulants of x and of y is *one and the same*, no matter what value of t we are using!

Idea of proof for formula (6).

For every $t \in \mathbb{R}$, consider the statement:

$$\text{(Statement } t\text{)} \quad \left\{ \begin{array}{c} \text{The formula (6) holds true for this } t \\ \text{and for any freely independent elements } x, y \text{ in} \\ \text{some non-commutative probability space } (\mathcal{A}, \varphi) \end{array} \right\}.$$

The action by conjugation of the u_q's on multiplicative functions allows us to prove the following fact:

Fact. *If there exists a $t_o \in \mathbb{R}$ for which (Statement t_o) is true, then it follows that (Statement t) is true for all $t \in \mathbb{R}$.*

But it is known since the 1990's that (Statement t_o) is true for $t_o = 1$ – basic description of multiplication of free random variables in terms of free cumulants. The above "Fact" then assures us that (Statement t) is indeed true for all t; in particular, at $t = 0$ we retrieve the result known for a while, about how multiplication of free random variables is described in terms of Boolean cumulants.

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Thank you for your attention!