Duality for optimal couplings in free probability

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Non-commutative probability spaces

A tracial W^*-algebra is a pair $\mathcal{A} = (A, \tau)$ with W^*-algebra A and a linear functional $\tau : A \to \mathbb{C}$ (trace) such that

- Unital: $\tau(1) = 1$,
- Positive: $\tau(a^* a) \geq 0$,
- Tracial: $\tau(ab) = \tau(ba)$,
- Faithful: $\tau(a^* a) = 0 \implies a = 0$.

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Every commutative tracial W^*-algebra is isomorphic to some $L^\infty(\Omega, P)$ with a trace $\tau(X) = \int X \, dP$.

Hence, a tracial W^*-algebra may be viewed as a non-commutative probability space.

Non-commutative probability spaces

classical	non-commutative
$L^\infty(\Omega, P)$	A
expectation $\mathbb{E}$	trace τ
bounded random variable	$X \in A$
bounded real random variable	$X \in A_{\mathrm{sa}}$ (self-adjoint)
bounded $\mathbb{R}^m$-valued random variable	$X = (X_1, \ldots, X_m) \in A_{\mathrm{sa}}^m$

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Let $L^2(A, \tau)$ be the GNS construction from (A, τ):

$$\langle a, b \rangle := \tau(a^* b).$$

Non-commutative laws

- Let $\Sigma_{m,R}$ be the collection of unital, positive, tracial linear maps $\mu : \mathbb{C}\langle X_1, \dots, X_m \rangle \to \mathbb{C}$ satisfying

$$|\mu(X_{i_1} \dots X_{i_n})| \leq R^n.$$

Each element in $\Sigma_{m,R}$ is called non-commutative law.
It encodes the non-commutative moments of some tuple of non-commutative random variables.

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- For $X = (X_1, \cdots, X_m) \in A_{\text{sa}}^m$, the associated non-commutative law is

$$\lambda_X(p) = \tau(p(X)).$$

- Tracial non-commutative laws $\leftrightarrow$ tracial W^*-algebras with a specified generating m-tuple up to isomorphism.

Goal of talk

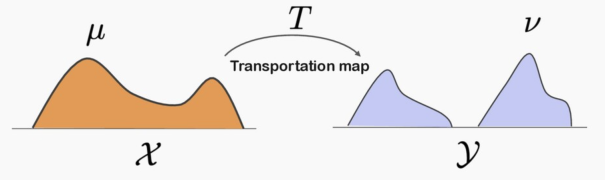
Optimal couplings of two non-commutative laws?

- Review on classical optimal transport theory
- Motivations of optimal couplings in non-commutative setting
- Duality for optimal couplings
- Geodesics via Hopf-Lax semigroup

Classical transportation theory

Monge's optimal transport problem: given $\mu, \nu \in \mathcal{P}(\mathbb{R}^m)$,

$$\min_{T_\# \mu = \nu} \int_{\mathbb{R}^m} |x - T(x)|^2 d\mu(x).$$



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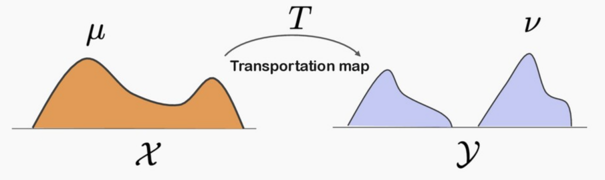

Monge problem can be ill-posed since

- Transport T may not exist.
- Constraint $T_{\#}\mu = \nu$ is not sequentially closed w.r.t. any reasonable weak topology.

Classical optimal couplings

Relaxation of Monge problem? Kantorovich (or Wasserstein) distance

$$d_W^2(\mu,\nu) := \inf_{(X,Y)\text{ coupling}} \mathbb{E}|X-Y|^2.$$

A coupling of μ and ν is a pair of random variables (X,Y) in some probability space such that $X \overset{\text{law}}{\sim} \mu$ and $Y \overset{\text{law}}{\sim} \nu$.

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Properties:

- Set of coupling is compact w.r.t. weak topology in $\mathcal{P}(\mathbb{R}^m \times \mathbb{R}^m)$.
- Minimizer (*optimal coupling*) always exists.
- If $\mu \in \mathcal{P}_2(\mathbb{R}^m)$ is absolutely continuous, then optimal coupling is induced by a gradient of convex function.
- d_W metrizes the weak topology on $\mathcal{P}_2(\mathbb{R}^m)$.

Non-commutative optimal couplings (Biane-Voiculescu)

Recall: Let $\Sigma_{m,R}$ be a collection of unital, positive, tracial linear maps $\mu : \mathbb{C}\langle X_1, \ldots, X_m \rangle \to \mathbb{C}$ satisfying $|\mu(X_{i_1} \ldots X_{i_n})| \leq R^n$.

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- A coupling of μ and ν is a tracial W^*-algebra $\mathcal{A} = (A, \tau)$ and random variables $X, Y \in A_{\text{sa}}^m$ with $\lambda_X = \mu$ and $\lambda_Y = \nu$.
- Wasserstein distance

$$d_W(\mu, \nu) := \inf_{(X,Y) \text{ coupling}} \|X - Y\|_{L^2(A,\tau)_{\text{sa}}^m}$$

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  Optimal couplings exist by compactness of $\Sigma_{2m,R}$.
- d_W defines a metric on $\Sigma_{m,R}$. It does not metrize the weak-$*$ topology (explained in David's talk).

Motivation

Transportation in non-commutative probability spaces.

Theorem (Guionnet-Shlyakhtenko, 14')

Consider a free Gibbs law μ_V with a log-density $V = \frac{1}{2}\sum_{i=1}^{m} X_i^2 + W$. For small enough W, there exists a transport map T from the law of semicircle family to μ_V.

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Theorem (Jekel-Li-Shlyakhtenko, 21')

Let μ be any non-commutative law. If $\nabla\phi$ is close to the identity map, then $\nabla\phi$ is an optimal transport from μ to $(\nabla\phi)_{\#}\mu$. In particular, transport map T above is optimal for W small enough.

Systematic theory for optimal transport in non-commutative probability spaces?

Classical Monge-Kantorovich duality

Given $X \overset{\text{law}}{\sim} \mu$ and $Y \overset{\text{law}}{\sim} \nu$,

$$\text{minimizing } \mathbb{E}|X - Y|^2 \iff \text{maximixing } \mathbb{E}\langle X, Y\rangle.$$

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Classical duality

$$\sup_{(X,Y) \text{ coupling}} \mathbb{E}\langle X, Y\rangle = \inf \Big(\int f \, d\mu + \int g \, d\nu \Big),$$

where inf is taken over pairs (f, g) of convex functions f, g satisfying $f(x) + g(y) \geq \langle x, y\rangle$.

Classical Monge-Kantorovich duality

Classical duality

Let (X, Y) be a coupling of μ and ν. Then, the following are equivalent.

1. (X, Y) is an optimal coupling.
2. There exist convex functions f, g with $f(x) + g(y) \geq \langle x, y \rangle$ everywhere such that

$$\mathbb{E}f(X) + \mathbb{E}g(Y) = \mathbb{E}\langle X, Y \rangle.$$

3. There exists a convex function f such that $Y \in \partial f(X)$.

For a convex function $f : H \to \mathbb{R}$, we say $y \in \partial f(x)$ (subgradient) if

$$f(x') - f(x) \geq \langle x' - x, y \rangle, \qquad \forall x' \in H.$$

Non-commutative case: tracial W^*-function

- In the classical setting, $(B(0,1), \mathrm{Leb}_{B(0,1)})$ is a universal probability space. In fact, any $\mu \in \mathcal{P}_2(\mathbb{R}^m)$ is a push-forward of $\mathrm{Leb}_{B(0,1)}$ under the gradient of convex function.

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- In the non-commutative setting, there is no separable tracial W^*-algebra that contains a copy of all others (Ozawa, 04'). Need to consider functions f defined for *all* tracial W^*-algebras.

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Definition

A tracial W^*-function is a collection $f^{\mathcal{A}} : L^2(\mathcal{A})^m_{\mathrm{sa}} \to (-\infty, \infty]$ of functions for each tracial W^*-algebra such that if $\iota : \mathcal{A} \to \mathcal{B}$ is a trace-preserving unital $*$-homomorphism, then $f^{\mathcal{A}} = f^{\mathcal{B}} \circ \iota$.

Convex functions

Convex functions on non-commutative probability spaces?

In the classical case, consider functions of random variables.
Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function and $(\Omega, \mathcal{F}, P)$ be a probability space.
Define $\tilde{f}(X) = \mathbb{E}f(X)$ for random variables $X \in L^2(\Omega, P; \mathbb{R}^m)$.

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Then,

- $\tilde{f}(X)$ depends only on the law of X.
- $\tilde{f}$ is convex.
- If $\mathcal{G}$ is a σ-subalgebra of $\mathcal{F}$, then

$$\tilde{f}(\mathbb{E}[X|\mathcal{G}]) \leq \tilde{f}(X).$$

E-convex functions

Definition

A tracial W^*-function $f = (f^{\mathcal{A}})_{\mathcal{A}}$ is said to be *E-convex* if

1. Each $f^{\mathcal{A}}$ is convex and lower semi-continuous on $L^2(\mathcal{A})^m_{\mathrm{sa}}$.
2. If $\iota : \mathcal{A} \to \mathcal{B}$ is a trace-preserving $*$-homomorphism inclusion and $E = \iota^* : \mathcal{B} \to \mathcal{A}$ is the corresponding trace-preserving conditional expectation, then $f^{\mathcal{A}}(E[X]) \leq f^{\mathcal{B}}(X)$.

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Lemma

Let f be a E-convex tracial W^*-function. Then, for any $\mathcal{A}$ and $X \in L^2(\mathcal{A})^m_{\text{sa}}$, there exists a subgradient $Y \in L^2(\mathrm{W}^*(X))^m_{\text{sa}}$ of $f^{\mathcal{A}}$ at X.

Non-commutative duality

For a non-commutative law μ and tracial W^*-function f, define

$$\mu(f) := f^{\mathcal{A}}(X),$$

where $(\mathcal{A}, X)$ is a GNS construction of μ.

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Proposition (GJNS)

We have the following duality:

$$\sup_{(X,Y)\text{ coupling}} \langle X, Y\rangle_{L^2(\mathcal{A})^m_{\mathrm{sa}}} = \inf \big(\mu(f) + \nu(g)\big),$$

where inf is taken over admissible pairs (f, g) of E-convex functions:

$$f^{\mathcal{A}}(X_1) + g^{\mathcal{A}}(X_2) \geq \langle X_1, X_2\rangle_{L^2(\mathcal{A})^m_{\mathrm{sa}}}, \qquad \forall X_1, X_2 \in L^2(\mathcal{A})^m_{\mathrm{sa}}.$$

Non-commutative duality

Proposition (GJNS)

Let $(\mathcal{A}, X, Y)$ be a coupling of μ and ν. The following are equivalent.

1. $(\mathcal{A}, X, Y)$ is an optimal coupling.
2. There exists an admissible pair (f, g) such that
$$f^{\mathcal{A}}(X) + g^{\mathcal{A}}(Y) = \langle X, Y \rangle_{L^2(\mathcal{A})^m_{\mathrm{sa}}}.$$
3. There exists an E-convex W^*-function f such that $Y \in \eth f^{\mathcal{A}}(X)$.

Inf-convolution

- Inf-convolution: for tracial W^*-functions f and g,

$$(f \square g)^{\mathcal{A}}(X) := \inf_{\iota:\mathcal{A}\to\mathcal{B}, Y \in L^2(\mathcal{B})_{\mathrm{sa}}^m} \left\{ f^{\mathcal{B}}(\iota(X) - Y) + g^{\mathcal{B}}(Y) \right\}.$$

$f \square g$ is a tracial W^*-function.

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  $f \square g$ is a tracial W^*-function.
- Example: $q_t^{\mathcal{A}}(X) = (1/2t)\|X\|^2_{L^2(\mathcal{A})^m_{\mathrm{sa}}}$. It satisfies $q_s \square q_t = q_{s+t}$ and

$$q_s \square (q_t \square f) = (q_s \square q_t) \square f = q_{s+t} \square f.$$

  $(q_t \square f)_{t>0}$ is a W^*-analog of Hopf-Lax semigroup:

$$\begin{cases} u_t + \frac{1}{2}\|\nabla u\|^2 = 0, \\ u(0, \cdot) = f. \end{cases}$$

Regularization of Hopf-Lax semigroup

Let $q_t^{\mathcal{A}}(X) = (1/2t)\|X\|^2_{L^2(\mathcal{A})^m_{\mathrm{sa}}}$ for $t > 0$.

Proposition (GJNS)

Let f be an E-convex function. Then, the following are equivalent.

1. $f = q_t \square g$ for some E-convex g.
2. $q_t - f$ is E-convex.
3. $\mathcal{L}f - q_{1/t}$ is E-convex.
4. $\eth f(X)$ consists of a single point $\nabla f(X) \in L^2(\mathrm{W}^*(X))^m_{\mathrm{sa}}$, and ∇f defines a $1/t$-Lipschitz function.

$\mathcal{L}f$ denotes Legendre transform:

$$\mathcal{L}f^{\mathcal{A}}(X) = \sup_{\iota:\mathcal{A}\to\mathcal{B},\, Y\in L^2(\mathcal{B})^m_{\mathrm{sa}}} \{\langle \iota(X), Y\rangle - f^{\mathcal{B}}(Y)\}.$$

Applications — isomorphism of von Neumann algebras

Let $(\mathcal{A}, X, Y)$ be an optimal coupling of $\mu, \nu \in \Sigma_{m,R}$.
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We have $\mathrm{W}^*((1-t)X + tY) = \mathrm{W}^*(X, Y)$ for $t \in (0,1)$.

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Theorem (GJNS)

We have $\mathrm{W}^*((1-t)X + tY) = \mathrm{W}^*(X, Y)$ for $t \in (0,1)$.

(Proof) $Y \in \eth f(X)$ for some E-convex f. Let

$$X_t = (1-t)X + tY, \qquad f_t(X) = (1-t)\frac{1}{2}\|X\|_{L^2}^2 + tf(X).$$

Then,

$$X_t \in \eth f_t(X) \Longrightarrow X \in \eth \mathcal{L} f_t(X_t).$$

$q_{1-t} - \mathcal{L} f_t$ is E-convex, which implies $X \in L^2(\mathrm{W}^*(X_t))_{\mathrm{sa}}^m$.

Thank you.