# Infinite dimensional phenomena in the analysis of Quantum Information Theory

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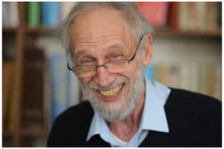


**The Connes Embedding Problem (CEP)**(Annals of Math, 1976):
Does every $\text{II}_1$-vN factor $(M, \tau)$ admit an embedding into

$$\mathcal{R}^\omega = \ell^\infty(\mathcal{R})/\{(T_n) : \lim_\omega \|T_n\|_2 = 0\},$$

$\omega =$ free ultrafilter on $\mathbb{N}$, $\|T\|_2 = \tau_{\mathcal{R}}(T^*T)^{1/2}$, $\tau_{\mathcal{R}} =$ trace on $\mathcal{R}$.

Roughly, does any $\text{II}_1$–vN factor *approximately embed* into a matrix alg?
Or, $\mathcal{R}$ We living in the Matrix? (Araiza-de Santiago, Notices AMS, 2016.)

**Theorem** (Kirchberg '93): Let $(M, \tau)$ be a $\text{II}_1$-vN factor with n.f. tracial state $\tau$. Then $M$ admits an embedding into $\mathcal{R}^\omega$ **iff** $\forall\ \varepsilon > 0$ and any set $u_1, \ldots, u_n$ of unitaries in $M$, $\exists\ k \geq 1$ and unitaries $v_1, \ldots, v_n$ in $M_k(\mathbb{C})$ :

$$\left|\tau(u_j^* u_i) - \mathrm{tr}_k(v_j^* v_i)\right| < \varepsilon, \qquad 1 \leq i, j \leq n.$$

Consider the following sets of $n \times n$ matrices of correlations, $n \geq 2$:

$$\begin{array}{rcl} \mathcal{G}_{\mathrm{matr}}(n) & = & \bigcup_{k\geq 1} \Big\{ \big[\mathrm{tr}_k(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in } M_k(\mathbb{C}) \Big\}, \\ \cap\!| & & \\ \mathcal{G}_{\mathrm{fin}}(n) & = & \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary} \\ & & \qquad \text{finite dimens C}^*\text{-alg } (\mathcal{A}, \tau) \Big\}, \\ \cap\!| & & \\ \mathcal{G}(n) & = & \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary} \\ & & \qquad \mathrm{II}_1\text{-vN factors } (M, \tau) \Big\}. \end{array}$$

All sets equal if $n = 2$.

Related: $D_{\mathrm{matr}}(n) \subseteq D_{\mathrm{fin}}(n) \subseteq D(n)$ where unitaries are replaced by proj.

**Theorem** (Kirchberg '93): The Connes Embedding Problem has a positive answer iff $\mathcal{G}(n) = \mathrm{cl}(\mathcal{G}_{\mathrm{matr}}(n))$, for all $n \geq 3$.

**Theorem** (M.-Rørdam '19):

- $\mathcal{G}_{\mathrm{matr}}(n)$ is neither convex, nor closed when $n \geq 3$.
- $\mathcal{G}_{\mathrm{fin}}(n)$ is convex for all $n \geq 2$, but not closed when $n \geq 11$.

**A Trick**: Let $p_1, \dots, p_n$ be projections in a vN alg $(M, \tau_M)$ with normal faithful tracial state. Define unitaries $u_0, u_1, \dots u_{2n} \in M$ by $u_0 = 1$ and

$$u_j = 2p_j - 1,\ 1 \le j \le n, \quad u_j = \frac{1}{\sqrt{2}}(u_{j-n} + i \cdot 1),\ n+1 \le j \le 2n.$$

Let $(N, \tau_N)$ be a vN alg with normal faithful tracial state. Then $\exists$ unitaries $v_0, v_1, \dots, v_{2n} \in N$ s.t. $\tau_N(v_j^* v_i) = \tau_M(u_j^* u_i), 0 \le i, j \le 2n$, **iff** $\exists$ projections $q_1, \dots, q_n \in N$ satisfying $\tau_N(q_j q_i) = \tau_M(p_j p_i), 1 \le i, j \le n$.

- Idea: the map $u_j \mapsto v_j$, extended linearly between Euclidean spaces $(\mathrm{Span}\{u_0, \dots u_{2n}\}, \langle \cdot, \cdot \rangle_{\tau_M})$, $(\mathrm{Span}\{v_0, \dots v_{2n}\}, \langle \cdot, \cdot \rangle_{\tau_N})$ is an isometry.

**Corollary**: If $[\tau_M(p_j p_i)] \in \overline{\mathcal{D}_{\mathrm{fin}}(n)} \setminus \mathcal{D}_{\mathrm{fin}}(n)$, then the corresponding $2n+1$ unitaries satisfy $[\tau_M(u_j^* u_i)] \in \overline{\mathcal{G}_{\mathrm{fin}}(2n+1)} \setminus \mathcal{G}_{\mathrm{fin}}(2n+1)$.

- This proves "$D_{\mathrm{fin}}(n)$ not closed $\Rightarrow \mathcal{G}_{\mathrm{fin}}(2n+1)$ not closed".
- To prove $\mathcal{G}_{\mathrm{matr}}(n)$ not closed, $n \ge 3$, note $D_{\mathrm{matr}}(n)$ not closed for $n \ge 1$ (since it's rationally convex, but not convex), and use the **Trick**.

To show $D_{\text{fin}}(n)$ not closed, $n \geq 5$, we streamline Dykema-Paulsen-Prakash '17. Let $\Sigma_n$ be the set of $\alpha \geq 0$ for which $\exists$ projections $p_1, \ldots, p_n$ on a Hilbert space $H$ such that $\sum_{j=1}^n p_j = \alpha \cdot I_H$. Known: $\Sigma_n \subset \mathbb{Q}$, when $n \leq 4$.

**Theorem** (Kruglyak-Rabanovich-Samoilenko '02): Let $n \geq 5$. There exist projections $p_1, \ldots, p_n$ on a *finite dimensional* Hilbert space $H$ s.t. $\sum_{j=1}^n p_j = \alpha \cdot I_H$ **iff** $\alpha \in \Sigma_n \cap \mathbb{Q}$. Furthermore,

$$\left[\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n})\right] \subseteq \Sigma_n.$$

**Theorem**: Let $n \geq 5$ and $t \in \left[\frac{1}{2}(1 - \sqrt{1 - 4/n}), \frac{1}{2}(1 + \sqrt{1 - 4/n})\right]$. Define $A_t^{(n)} = [A_t^{(n)}(i,j)]_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$ by

$$A_t^{(n)}(i,i) = t, \quad A_t^{(n)}(i,j) = \frac{t(nt-1)}{n-1}, i \neq j.$$

If $t \notin \mathbb{Q}$, then $A_t^{(n)} \in \overline{D_{\text{fin}}(n)} \setminus D_{\text{fin}}(n)$. Hence $D_{\text{fin}}(n)$ not closed.

A quantum channel is a completely positive $\text{tr}_n$-preserving map on $M_n(\mathbb{C})$.

**Definition/Theorem** (Haagerup-M. '11): A unital quantum channel $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is factorizable iff $\exists$ $(N, \tau_N)$ vN algebra with n.f. tracial state (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$ s.t.

$$Tx = (\text{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in M_n(\mathbb{C}).$$

**Examples**: 1) $T(x) = u^*xu$, $x \in M_n(\mathbb{C})$, for some $u \in M_n(\mathbb{C})$ unitary.

(Kümmerer '83): Any unital qubit ($n = 2$) is a convex combination of unitarily implemented channels.

2) *Completely depolarizing channel* $S_n(x) = \text{tr}_n(x)1_n$, $x \in M_n(\mathbb{C})$.

3) Schur product channels $T_B$ associated to real correlation matrices $B$.

- The set of factorizable channels on $M_n(\mathbb{C})$ is convex and closed, $n \geq 2$.

**Theorem** (Haagerup-M. '11): For all $n \geq 3$, there exist non-factorizable quantum channels on $M_n(\mathbb{C})$. Each such channel violates the Asymptotic Quantum Birkhoff Conjecture of Smolin-Verstraete-Winter '05.

**Question**: Do we need inf dimens vN algs to describe factoriz channels?

▶ For a factorizable channel, *minimal* ancilla (and its size) **not** unique. E.g., possible ancillas for $S_n$ are: $\mathbb{C}^{n^2}$, $M_n(\mathbb{C})$, but also (a corner of) the reduced free product vN alg $(M_n(\mathbb{C}), \mathrm{tr}_n) * (M_n(\mathbb{C}), \mathrm{tr}_n)$.

For $n \geq 2$, let $\mathcal{FM}(n)$ denote all factorizable quantum channels on $M_n(\mathbb{C})$. Denote further by $\mathcal{FM}_{\mathrm{fin}}(n)$ those admitting a finite dim ancilla.

**Theorem** (Haagerup-M. '15) The Connes Embedding Problem has a positive answer iff $\mathcal{FM}_{\mathrm{fin}}(n)$ is dense in $\mathcal{FM}(n)$, for all $n \geq 3$.

**Theorem** (M.-Rørdam '19): $\mathcal{FM}_{\mathrm{fin}}(n)$ is not closed, whenever $n \geq 11$. For each such $n$, $\exists$ factorizable quantum channels on $M_n(\mathbb{C})$ which **do** require infinite dimens (even type $\mathrm{II}_1$) ancilla.

▶ (Haagerup-M '11): A Schur multiplier $T_B$ is factorizable iff $B \in \mathcal{G}(n)$ (i.e., $B = [\tau(u_j^* u_i)]$, $u_1, \ldots u_n$ unitaries in a $\mathrm{II}_1$- factor $(M, \tau)$). Further, $T_B \in \mathcal{FM}_{\mathrm{fin}}(n)$ iff $B \in \mathcal{G}_{\mathrm{fin}}(n)$.

(Rørdam-M '20): A new view-point on factorizable channels, leading to further connections (and interesting open problems in C*-algebras):

- ▶ $\mathcal{FM}(n)$ is *parametrized by* simplex of tracial states $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

More precisely, if $\tau \in T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, let

$$C_\tau(i,j;k,\ell) = n\tau\big(\iota_2(e_{k\ell})^* \, \iota_1(e_{ij})\big), \qquad 1 \le i,j,k,\ell \le n,$$

where $\iota_1, \iota_2 \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) * M_n(\mathbb{C})$ are the canonical inclusions. Then $C_\tau \in M_{n^2}(\mathbb{C})$ is positive, hence it is the Choi matrix of some c.p. lin map $T_\tau : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, which turns out to be a factoriz quantum channel!

In fact, the map $\Phi\colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto \Phi(\tau) := T_\tau$ is an affine continuous surjection, satisfying, moreover,

$$\Phi(T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\rm fin}(n),$$

where $T_{\rm fin}$ = tracial states that factor through finite dim. $C^*$-alg.

The affine cont surj $\Phi\colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto T_\tau$, satisfies

- $\Phi(T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\rm fin}(n)$,
- $\Phi(\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}) = \overline{\mathcal{FM}_{\rm fin}(n)}$,

where $T_{\rm fin}$ = tracial states that factor through finite dim. $C^*$-alg.

**Thm** (Haagerup-M '15) Connes Embedding Problem (CEP) has positive answer **iff** $\overline{\mathcal{FM}_{\text{fin}}(n)} = \mathcal{FM}(n)$, $\forall n \geq 3$.

**Question**: What can we say about $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}$ ?

- (Exel–Loring '92): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ residually finite dim. (RFD)
- (Blackadar '85): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ semi-projective.

In general, given $A =$ (sep) unital tracial $C^*$-algebra, we have inclusions:

$$T_{\rm fin}(A) \subseteq \overline{T_{\rm fin}(A)} \subseteq T_{\rm qd}(A) \subseteq T_{\rm am}(A) \subseteq T_{\rm hyp}(A) \subseteq T(A),$$

where $T_{\rm qd}(A) =$ quasi-diagonal traces, $T_{\rm am}(A) =$ amenable (liftable) traces, $T_{\rm hyp}(A) =$ hyperlinear traces (i.e., traces $\tau$ st $\pi_\tau(A)'' \hookrightarrow \mathcal{R}^\omega$).

**Reformulation of CEP:** For all sep. unital tracial $C^*$-algs $(A, \tau)$, there is a unital trace- preserving $^*$-hom $\varphi \colon A \to \prod_{n=1}^\infty M_{k_n}/I^\omega$, for some $k_n \geq 1$.

- CEP pos answer **iff** $T_{\rm hyp}(A) = T(A)$, for all $C^*$-alg $A$.
- (N. Brown '06): $\exists$ exact RFD $C^*$-alg $A$ s.t. $T_{\rm am}(A) \neq T_{\rm hyp}(A)$.
- Open if $T_{\rm qd}(A) = T_{\rm am}(A)$. Strong pos results: Tikuisis-Winter-White, Schafhauser, Gabe.
- $A$ (weakly) semi-projective $\implies$ $\overline{T_{\rm fin}(A)} = T_{\rm qd}(A)$
- (Hadwin–Shulman '17): $\exists$ RFD $C^*$-alg $A$ s.t. $\overline{T_{\rm fin}(A)} \neq T_{\rm qd}(A)$.

**Thm** (Rørdam-M '20): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

**Thm** (Rørdam-M): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

**Cor:** CEP pos **iff** $T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C})) = T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $\forall\, n \geq 3$.

**Further results**: Let $A$ be a unital $C^*$-algebra.

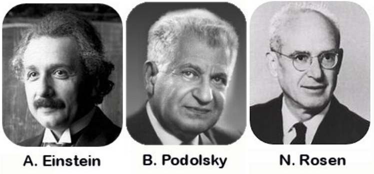
- If $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, then $A$ generated by $n^2$ elem.
- If $A$ gen by $n-1$ unitaries, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$.

As unital separable $\mathcal{Z}$-stable C*-alg are singly generated (Thiel-Winter), we deduce: If $A$ is a simple unital inf dim AF-algebra, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, $n \geq 3$.

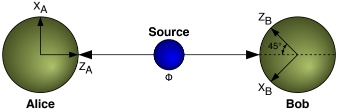
**Thm** (Rørdam-M): Each metrizable Choquet simplex is affinely homeo to a face of $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $n \geq 3$.

**(Open) question**: Is $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$ the Poulsen simplex?

# Quantum Correlations and The Einstein–Podolsky–Rosen paradox




Alice and Bob, residing in spatially separated labs, each receives (part of) a quantum system on which they can perform measurements.




Let's say that Alice and Bob can measure any one of $n$ possible observables, each with $k$ possible outcomes. Let

$$P(a, b \mid x, y)$$

be the probability that Alice gets outcome $a$ and Bob outcome $b$, when Alice measures observable $x$ and Bob measures observable $y$.

**Hidden variables - the classical model:** $\exists$ prob. space $(\Omega, \mu)$ and partitions $\{A_a^x\}_a$ and $\{B_b^y\}_b$ of $\Omega$ (one for each $x, y$) s.t.

$$P(a, b \mid x, y) = \mu(A_a^x \cap B_b^y).$$

**Definition:** A PVM (projection valued measure) is a $k$-tuple $P_1, \ldots, P_k$ of projections on a Hilbert space $H$ s.t. $\sum_{j=1}^k P_j = I$.

**Two quantum models** for interpreting the physical separation:

*Tensor product:* $\exists$ Hilbert spaces $H_A$, $H_B$, PVMs $\{P_a^x\}_a$ on $H_A$, $\{Q_b^y\}_b$ on $H_B$, and unit vector $\psi \in H_A \otimes H_B$ s.t.

$$P(a, b \mid x, y) = \langle (P_a^x \otimes Q_b^y)\psi, \psi \rangle.$$

*Commutativity:* $\exists$ Hilbert space $H$, *commuting* PVMs $\{P_a^x\}_a, \{Q_b^y\}_b$ on $H$, and unit vector $\psi \in H$ s.t.

$$P(a, b \mid x, y) = \langle P_a^x Q_b^y \psi, \psi \rangle.$$

We obtain following convex sets of $nk \times nk$ real matrices, rows are indexed by $(a,x)$, columns by $(b,y)$:

$$\mathcal{C}_c(n,k) = \Big\{ \Big[ \mu(A_a^x \cap B_b^y) \Big] : \{A_a^x\}_a, \{B_b^y\}_b \text{ partitions of } (\Omega,\mu) \Big\},$$

$$\mathcal{C}_{qs}(n,k) = \Big\{ \Big[ \langle (P_a^x \otimes Q_b^y)\psi, \psi \rangle \Big] : \{P_a^x\}_a, \{Q_b^y\}_b \text{ PVMs}, \ \psi \in H_A \otimes H_B \Big\},$$

$$\mathcal{C}_{qc}(n,k) = \Big\{ \Big[ \langle P_a^x Q_b^y \psi, \psi \rangle \Big] : \{P_a^x\}_a, \{Q_b^y\}_b \text{ PVMs}, [P_a^x, Q_b^y] = 0, \psi \in H \Big\}.$$

Let $\mathcal{C}_{qs}^{\mathrm{fin}}(n,k)$ and $\mathcal{C}_{qc}^{\mathrm{fin}}(n,k)$ further denote the correlation sets where the Hilbert spaces $H_A, H_B$, resp., $H$ are *finite dimensional*.

$$\begin{array}{ccccccccc} & & \mathcal{C}_{qs}^{\mathrm{fin}}(n,k) & & \overset{\text{Tsirelson}}{=} & & \mathcal{C}_{qc}^{\mathrm{fin}}(n,k) & & \\ & & \cap\!\vert & & & & \cap\!\vert & & \\ \mathcal{C}_c(n,k) & \subseteq & \mathcal{C}_{qs}(n,k) & \subseteq & \mathrm{cl}(\mathcal{C}_{qs}(n,k)) & \subseteq & \mathcal{C}_{qc}(n,k) & \subseteq & M_{nk}(\mathbb{R}) \end{array}$$

- $\mathrm{cl}(\mathcal{C}_{qs}^{\mathrm{fin}}(n,k)) = \mathrm{cl}(\mathcal{C}_{qs}(n,k))$.

$$
\begin{array}{ccccccccc}
 & & \mathcal{C}_{qs}^{\text{fin}}(n,k) & & \overset{\text{Tsirelson}}{=} & & \mathcal{C}_{qc}^{\text{fin}}(n,k) & & \\
 & & \cap\!\mid & & & & \cap\!\mid & & \\
\mathcal{C}_c(n,k) & \subseteq & \mathcal{C}_{qs}(n,k) & \subseteq & \mathrm{cl}(\mathcal{C}_{qs}(n,k)) & \subseteq & \mathcal{C}_{qc}(n,k) & \subseteq & M_{nk}(\mathbb{R})
\end{array}
$$

- $\mathrm{cl}(\mathcal{C}_{qs}^{\text{fin}}(n,k)) = \mathrm{cl}(\mathcal{C}_{qs}(n,k))$.

Bell's ineq: $\mathcal{C}_c(n,k) \neq \mathcal{C}_{qs}(n,k)$. Also follows from Grothendieck's Thm!

**Conjecture/Problem** (Tsirelson): $\mathrm{cl}(\mathcal{C}_{qs}(n,k)) \overset{?}{=} \mathcal{C}_{qc}(n,k)$.

**Breakthrough** (Slofstra '16, '17): $\mathcal{C}_{qs}(n,k) \neq \mathcal{C}_{qc}(n,k)$, and furthermore, $\mathcal{C}_{qs}(n,k)$ is **not** closed, for $n, k$ large.

(Dykema-Paulsen-Prakash '17), (M.-Rørdam '19): $\mathcal{C}_{qs}(5,2)$ **not** closed.

## Some connections to the analysis of (synchronous) quantum corr and CEP

- $\mathbb{F}(n,k) = \mathbb{Z}_k * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$, $n$ copies, $n, k \geq 2$.
- $C^*(\mathbb{F}(n,k)) = C^*(q_{j,x} \mid q_{j,x} = q_{j,x}^* = q_{j,x}^2,\ \sum_{j=1}^k q_{j,x} = 1)$.

**Definition**: A "correlation" $[(p(i,j \mid x,y)]$ is *synchronous* if $\forall\, 1 \leq x \leq n$, $p(i,j \mid x,x) = 0$ whenever $i \neq j$.

**Theorem** (PSSTW '16): For $n, k \geq 2$,

$$\begin{aligned} C_{qc}^s(n,k) &= \left\{ \left[\tau(q_{j,x} q_{i,y})\right]_{(i,x;j,y)} \mid \tau \in T(C^*(\mathbb{F}(n,k))) \right\} \\ C_q^s(n,k) &= \left\{ \left[\tau(q_{j,x} q_{i,y})\right]_{(i,x;j,y)} \mid \tau \in T_{\text{fin}}(C^*(\mathbb{F}(n,k))) \right\}. \end{aligned}$$

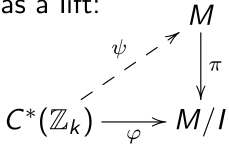
**Proposition** (Schafhauser, AIM '21): For all $n, k \geq 2$,

$$\overline{T_{\text{fin}}(C^*(\mathbb{F}(n,k)))} = T_{\text{hyp}}(C^*(\mathbb{F}(n,k))).$$

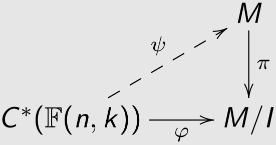
In particular, $\overline{T_{\text{fin}}(C^*(\mathbb{F}(n,k)))} = T(C^*(\mathbb{F}(n,k)))$ if CEP is true.

**Lemma** (Folklore): Let $I \lhd M$, where $M =$ unital $C^*$-alg of real rank zero (e.g., $M$ a vN algebra), and let $\pi\colon M \to M/I$ be the quotient mapping. If $q_1, \dots, q_k \in M/I$ are proj s.t. $\sum_{j=1}^k q_j = 1$, then $\exists\, p_1, \dots, p_k \in M$ proj with $\sum_{j=1}^k p_j = 1$ and $\pi(p_j) = q_j$.

Since $C^*(\mathbb{Z}_k)$ is generated by $k$ projections summing up to 1, each unital $^*$-hom $\varphi\colon C^*(\mathbb{Z}_k) \to M/I$ has a lift:




**Corollary:** Let $I \lhd M$, $\pi\colon M \to M/I$ as above. Then each unital $^*$-hom $\varphi\colon C^*(\mathbb{F}(n,k)) \to M/I$ lifts to a unital $^*$-hom:



**Proof of Prop**: Let $\tau \in T_{\mathrm{hyp}}(C^*(\mathbb{F}(n,k)))$. Then $\exists$ $^*$-hom $\varphi$:

$$\begin{array}{ccc} & & \prod_{n=1}^{\infty} M_{k_n} \\ & \overset{\psi}{\nearrow} & \downarrow \pi \\ C^*(\mathbb{F}(n,k)) & \xrightarrow[\varphi]{} & \prod_{n=1}^{\infty} M_{k_n}/I^{\omega} \end{array}$$

s.t. $\tau = \tau_\omega \circ \varphi$. Let $\psi$ be a lift of $\phi$ (by Corollary). Write $\psi = (\psi_n)_{n \geq 1}$, where $\psi_n \colon C^*(\mathbb{F}(n,k)) \to M_{k_n}$ unital $^*$-homs.

By definition of $\tau_\omega$, for all $a \in C^*(\mathbb{F}(n,k))$ we have

$$\tau(a) = (\tau_\omega \circ \varphi)(a) = \lim_{n \to \omega} (\mathrm{tr}_{k_n} \circ \psi_n)(a)$$

and $\mathrm{tr}_{k_n} \circ \psi_n \in T_{\mathrm{fin}}(C^*(\mathbb{F}(n,k)))$, showing that $\tau \in \overline{T_{\mathrm{fin}}(C^*(\mathbb{F}(n,k)))}$. □

**Theorem** (Kirchberg '93, Fritz/Junge et al '09, Ozawa '12): TFAE:

(i) $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$.

(ii) The Connes Embedding Problem has a positive answer.

(iii) Tsirelson's Conjecture is true: $\mathrm{cl}(\mathcal{C}_{qs}(n,k)) = \mathcal{C}_{qc}(n,k)$, $\forall n, k \geq 2$.

**Posted on arXiv**, Jan, 2020: $MIP^* = RE$, Ji, Natarajan, Vidick, Wright, Yuen, 165 pp.

Proving that the complexity class MIP* (quantum version of complexity class MIP=languages with a Multiprover Interactive Proof) contains an undecidable language, they conclude that Tsirelson's Conjecture is **false**!

- New version (with corrections) 206 pp., posted on arXiv, Sept, 2020.

These papers led to highly interesting interconnections between Op Alg, Group Theory, QIT, Computer Science, incl. *stability of groups, non-local games, self-testing*, etc.