Rationality for operators obtained from free semicircular elements

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Recent works related to rationality in free probability

In free probability, there are several works which deal with NC rational functions, $x_1x_2^{-1}(x_1+x_3)^{-1}, x_1^2x_2x_3^{-1}, x_1(1+x_2)^{-1}, \ldots$

T. Mai, R. Speicher and S. Yin.
The free field: realization via unbounded operators and Atiyah property.
arXiv:1905.08187, 2019.

B. Collins, T. Mai, A. Miyagawa, F. Parraud and S. Yin.
Convergence for noncommutative rational functions evaluated in random matrices.
arXiv:2103.05962, 2021.

O. Arizmendi, G. Cébron, R. Speicher, and Sheng. Yin.
Universality of free random variables: atoms for non-commutative rational functions
arXiv:2107.11507, 2021.

Hankel operator

Definition

Let $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{C}$. $H \in B(l^2(\mathbb{N}_{\geq 0}))$ is called the *Hankel operator* w.r.t. $\{\alpha_n\}_{n=0}^{\infty}$ if H satisfies

$$\langle He_m, e_n \rangle = \alpha_{m+n}.$$

Example

$H = PM_hU \in B(H_2)$ for $h \in L^{\infty}(S^1)$ is Hankel w.r.t $\{\hat{h}(n)\}_{n=0}^{\infty}$ where

- P: orthogonal proj onto the Hardy space $H_2 := \{f \in L^2(S^1) \mid \hat{f}(n) = 0, \ n \in \mathbb{Z}_{<0}\}$
- M_h: multiplication operator of h
- U: unitary operator s.t. $U(z^n) = z^{-n}$.

Kronecker's theorem

Fourier expansion of f $\to$ Hankel operator $\to$ some information about f.

Theorem (L. Kronecker, 1881.)

Let $a = \sum_{n=0}^{\infty} \alpha_n z^{-n-1} \in L^\infty(S^1)$. *Then we have*

$$\begin{aligned} & \text{Hankel operator w.r.t } \{\alpha_n\}_{n=0}^{\infty} \text{ has finite rank} \\ \Leftrightarrow \quad & a \text{ is rational (i.e. } {}^{\exists} p(z), q(z) \in \mathbb{C}[z] \text{ s.t. } a = p(z)/q(z)). \end{aligned}$$

In this case, the poles of a *are contained in* $\{|z| < 1\}$.

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Corollary (cf. the book "Noncommutative geometry")

For $a \in L^\infty(S^1)$,

$$[P, a] \text{ has finite rank} \Leftrightarrow a \text{ is rational.}$$

Division closure and rational closure

Let $\mathcal{A}$ be a unital algebra and $\mathcal{B} \subset \mathcal{A}$ be a unital subalgebra of $\mathcal{A}$.

Definition (Division closure)

The *division closure* of $\mathcal{B}$ in $\mathcal{A}$ is the smallest unital subalgebra $\mathcal{C}$ of $\mathcal{A}$ s.t. $\mathcal{B} \subset \mathcal{C}$ and

$$x \in \mathcal{C} \text{ is invertible in } \mathcal{A} \implies x^{-1} \in \mathcal{C}.$$

Definition (Rational closure)

The *rational closure* of $\mathcal{B}$ in $\mathcal{A}$ is the smallest unital subalgebra D of $\mathcal{A}$ s.t. $\mathcal{B} \subset \mathcal{D}$ and for any $n \in \mathbb{N}$.

$$X \in M_n(\mathcal{D}) \text{ is invertible in } M_n(\mathcal{A}) \implies X^{-1} \in M_n(\mathcal{D}).$$

From NC geometry - free group case

Noncommutative version of Kronecker's theorem was considered in terms of the quantized calculus in NC geometry.

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- ϕ can be extended to the unitary operator $P : l^2(\mathbb{F}_d) \to l^2(E) \oplus \mathbb{C}$.
- Consider a self-adjoint unitary operator

$$F := \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix} \in B(l^2(\mathbb{F}_d) \oplus l^2(E) \oplus \mathbb{C}).$$

Criterion of rationality for reduced free group C^*-algebra

- $C_{\mathrm{div}}(\mathbb{F}_d)$: division closure of $\mathbb{C}[\mathbb{F}_d]$ in $C^*_{\mathrm{red}}(\mathbb{F}_d)$
- $C_{\mathrm{rat}}(\mathbb{F}_d)$: rational closure of $\mathbb{C}[\mathbb{F}_d]$ in $C^*_{\mathrm{red}}(\mathbb{F}_d)$.

Theorem (G. Duchamp and C. Reutenauer, 1997)

For $a \in C^*_{\mathrm{red}}(\mathbb{F}_d)$,

$$[F, a] \text{ is a finite rank operator} \quad \Leftrightarrow \quad a \in C_{\mathrm{div}}(\mathbb{F}_d)$$
$$\Leftrightarrow \quad a \in C_{\mathrm{rat}}(\mathbb{F}_d).$$

This theorem was conjectured by A. Connes and solved by G. Duchamp and C. Reutenauer (*Invent. math.*, 1997).

From free probability - Full Fock space

- H : d-dimensional Hilbert space with o.n.b. $e_1, \ldots, e_d$.
- *Full Fock space* :

$$\mathcal{F}(H) := \bigoplus_{n=0}^{\infty} H^{\otimes n}$$

where $H^{\otimes 0} = \mathbb{C}\Omega$ with the vacuum vector Ω.

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Let $f \in H$.

- *Left creation operator* $l(f) \in B(\mathcal{F}(H))$:

$$l(f)\xi = f \otimes \xi, \; \xi \in H^{\otimes n}$$

- *Right creation operator* $r(f) \in B(\mathcal{F}(H))$:

$$r(f)\xi = \xi \otimes f, \; \xi \in H^{\otimes n}.$$

Free semicircular elements

- Put $l_i := l(e_i), r_i := r(e_i)$ and $s_i := l_i + l_i^*$.
- $L^\infty(\mathbf{s})$: v.N.algebra generated by $\mathbf{s} = (s_1, \ldots, s_d)$.

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Remark (e.g. Voiculescu, Dykema, Nica, 1992)

- Ω is a cyclic, separating and tracial vector for $L^\infty(\mathbf{s})$.
- Operators $s_1, \dots, s_d$ have *freely independent semicircle distribution* w.r.t. $\tau_\Omega(\cdot) = \langle \cdot \Omega, \Omega \rangle$

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- $C_{\mathrm{div}}(\mathbf{s})$: division closure of $\mathbb{C}\langle \mathbf{s} \rangle$ in $L^\infty(\mathbf{s})$.
- $C_{\mathrm{rat}}(\mathbf{s})$: rational closure of $\mathbb{C}\langle \mathbf{s} \rangle$ in $L^\infty(\mathbf{s})$.

Main result

Theorem (M. 2021)

For $a \in L^\infty(\mathbf{s})$, we have

$$\begin{aligned} \{[r_i^*, a]\}_{i=1}^{d} \text{ are finite rank operators } \quad &\Leftrightarrow \quad a \in C_{\mathrm{div}}(\mathbf{s}) \\ &\Leftrightarrow \quad a \in C_{\mathrm{rat}}(\mathbf{s}). \end{aligned}$$

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Remark

1. $[r_i, a] = -[r_i^*, a]$ for $a \in L^\infty(\mathbf{s})$.
2. $[r_i^*, s_j] = \delta_{i,j} P_\Omega$ where P_Ω is the orthogonal proj. onto $\mathbb{C}\Omega$ (Dual system), cf. Voiculescu, 1998.
3. By Leibniz rule, existence of a dual system immediately implies,

$$a \in \mathbb{C}\langle \mathbf{s} \rangle \implies \{[r_i^*, a]\}_{i=1}^d \text{ are finite rank ops}$$

The strategy of the proof

"$a \in C_{\mathrm{rat}}(\mathbf{s}) \implies \{[r_i^*, a]\}_{i=1}^{d}$ are finite rank ops" also follows from Leibniz rule (note $[r_i^*, a^{-1}] = -a^{-1}[r_i^*, a]a^{-1}$).

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Converse direction follows from the following strategy:

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2. NC formal series $\sum_{v \in [d]^*} \alpha_v X^v$ is rational by using the fundamental theorem.

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3. Matrix trick.
4. Estimation of operator norm (Haagerup's inequality).

Step 1. Fourier expansion

$[d] = \{1, \dots, d\}$: a set of letters. $[d]^*$: a set of words of $[d]$.
$\Omega \in [d]^*$: empty word.

- $\{e_w\}_{w \in [d]^*}$ is o.n.b. of $\mathcal{F}(H)$ where $e_{w_1 \cdots w_n} := e_{w_1} \otimes \cdots \otimes e_{w_n}$ and $e_\Omega = \Omega$.

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- $U_w := U_{k_1}(s_{i_1}) \cdots U_{k_n}(s_{i_n})$ where $w = i_1^{k_1} \cdots i_n^{k_n}$ $(i_1 \neq i_2 \neq \cdots \neq i_n)$ and $U_m(x)$ is the Chebyshev polynomial of the second kind and $U_\Omega = 1$.

- We can check $U_w \Omega = e_w$.

- (Fourier expansion)

$$\hat{a} = \sum_{w \in [d]^*} \alpha_w e_w = \sum_{w \in [d]^*} \alpha_w \hat{U}_w.$$

Step 1. Fourier expansion

- Let $\{0\}$ be a new letter.
- For $u \in [d]^*$, $v \in [d]^*$, we define

$$uv^{-1} = \begin{cases} w \text{ if } u = wv \\ 0 \text{ otherwise.} \end{cases}$$

- We set $U_0 = 0$.

Lemma (M. 2021)

For any $v, w \in [d]^*$ and $i \in [d]$,

$$[r_i^*, U_v]\hat{U}_w = \hat{U}_{v(iw^*)^{-1}}$$

where $w^* = w_n \cdots w_1$ for $w = w_1 \cdots w_n$.

Step 2. The fundamental theorem for NC rational series

From the previous lemma, for $a \in L^\infty(\mathbf{s})$ with $\hat{a} = \sum_{v \in [d]^*} \alpha_v \hat{U}_v$, $\{[r_i^*, a]\}_{i=1}^d$ are finite rank operators $\implies \mathrm{span}\left\{ \sum_{v \in [d]^*} \alpha_v \hat{U}_{vw^{-1}} \ : \ w \in [d]^* \right\}$ is finite dimensional.

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We can use the following theorem for $\sum_{v\in[d]^*} \alpha_v X^v$.

Theorem (Fliess, Jacobi, Kleene, Schützenberger,...)

Let $\sum_{v\in[d]^*} \alpha_v X^v$ be a NC formal power series, then TFAE:

1. $\operatorname{span}\left\{\sum_{v\in[d]^*} \alpha_v X^{vw^{-1}} \; : \; w \in [d]^*\right\}$ *is finite dim.*
2. *(Rational) $\sum_{v\in[d]^*} \alpha_v X^v$ belongs to the division closure of NC polynomials.*
3. *(Recognizable) There exists some $\lambda, \gamma \in \mathbb{C}^n$ and morphism $\mu : [d]^* \to M_n(\mathbb{C})$ s.t. $\alpha_v = {}^t\lambda\mu(v)\gamma$.*

Example of NC rational power series

Let us consider $S = \sum_{w \in [2]^*} 2^{-|w|} X^{w1} + \sum_{w \in [2]^*} 3^{-|w|} X^{w2}$.

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$$\mu(1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \ \mu(2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{3} \end{pmatrix},$$

Then we put $\lambda = {}^t \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$ and $\gamma = {}^t \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

$$S = \sum_{w \in [2]^*} {}^t \lambda \mu(w) \gamma X^w.$$

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$$S = \sum_{w \in [2]^*} {}^t\lambda \mu(w) \gamma X^w.$$

Note that we can formally compute as follows

$${}^t\lambda[1 - \mu(1)X_1 - \mu(2)X_2]^{-1}\gamma = {}^t\lambda \sum_{m=0}^{\infty} [\mu(1)X_1 + \mu(2)X_2]^m \gamma$$

$$= {}^t\lambda \sum_{m=0}^{\infty} \sum_{|w|=m} \mu(w) X^w \gamma = \sum_{w \in [2]^*} {}^t\lambda \mu(w) \gamma X^w = S$$

Step 3. Matrix trick

We can see $\{[r_i^*, a]\}_{i=1}^d$ are finite rank operators

$\implies \text{span}\left\{\sum_{v\in[d]^*} \alpha_v \hat{U}_{vw^{-1}} \ : \ w \in [d]^*\right\}$ is finite dim.

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Lemma (M. 2021)

For each $i \in [d]$, we put

$$S_i = E_{ii} \otimes \begin{pmatrix} s_i & -1 \\ 1 & 0 \end{pmatrix} + \sum_{j\neq i} E_{ji} \otimes \begin{pmatrix} s_i & -1 \\ 0 & 0 \end{pmatrix} \in M_d(\mathbb{C}) \otimes M_2(L^\infty(\mathbf{s})).$$

Then we have

$$U_v = \begin{pmatrix} 1 & 0 \end{pmatrix} ({}^t e_1 \otimes I_2)\ S^v\ (e \otimes I_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where we put $e = \sum_{i=1}^d e_i$.

Example of matrix trick: $d = 2$, $|v| = 2$

$$S_1 = \begin{pmatrix} s_1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ s_1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & 0 & s_2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & s_2 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then we have

$$S^{11} = \begin{pmatrix} \color{red}{U_{11}} & -U_1 & 0 & 0 \\ U_1 & -U_\Omega & 0 & 0 \\ U_{11} & -U_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, S^{12} = \begin{pmatrix} 0 & 0 & \color{red}{U_{12}} & -U_1 \\ 0 & 0 & U_2 & -U_\Omega \\ 0 & 0 & U_{12} & -U_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S^{21} = \begin{pmatrix} \color{red}{U_{21}} & -U_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ U_{21} & -U_2 & 0 & 0 \\ U_1 & -U_\Omega & 0 & 0 \end{pmatrix}, S^{22} = \begin{pmatrix} 0 & 0 & \color{red}{U_{22}} & -U_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & U_{22} & -U_2 \\ 0 & 0 & U_2 & -U_\Omega. \end{pmatrix}$$

Step 4. Estimation of operator norm

We want to show the convergence $\sum_{v\in[d]^*} \alpha_v S^v$ in operator norm.

Lemma (M. Bożėjko, 1998.)

For any $m \in \mathbb{N}_{\geq 0}$, we have

$$\|\sum_{|v|=m} \alpha_v U_v\| \leq (m+1)\sqrt{\sum_{|v|=m} |\alpha_v|^2}.$$

By using this inequality, we have

Lemma (M. 2021)

For any $m \in \mathbb{N}_{\geq 0}$, we have

$$\|\sum_{|v|=m} \alpha_v S^v\| \leq 4d^2(m+1)\sqrt{\sum_{|v|=m} |\alpha_v|^2}.$$

Estimation of operator norm

Indeed, we have

$$\sum_{|v|=m} \alpha_v S^v = \sum_{i,j\in[d]} E_{ij} \otimes \begin{pmatrix} \sum_{|v|=m-1} \alpha_{vj} U_{vj} & -\sum_{|v|=m-1} \alpha_{vj} U_v \\ \sum_{|v|=m-2} \alpha_{ivj} U_{vj} & -\sum_{|v|=m-2} \alpha_{ivj} U_v \end{pmatrix},$$

and thus operator norms of all entries are bounded by $(m+1)\sqrt{\sum_{|v|=m} |\alpha_v|^2}$.

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and thus operator norms of all entries are bounded by $(m+1)\sqrt{\sum_{|v|=m} |\alpha_v|^2}$.
Moreover, Kronecker's theorem tells us that if $\sum_{v\in[d]^*} |\alpha_v|^2 < \infty$ and $\sum_{v\in[d]^*} \alpha_v X^v$ is rational, then we have

$$\sqrt{\sum_{|v|=m} |\alpha_v|^2} \leq Mc^m$$

for some $M > 0$ and $0 < c < 1$.

Estimation of operator norm

By combining these results, we have

$$\|\sum_{|v|=m} \alpha_v S^v\| \leq 4d^2(m+1)Mc^m.$$

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Finally, we want to estimate $\| \sum_{|v|=m} \mu(v) S^v \|$ $(\alpha_v = {}^t\lambda\mu(v)\gamma)$.

Estimation of operator norm

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$$\|\sum_{|v|=m} \alpha_v S^v\| \leq 4d^2(m+1)Mc^m.$$

Finally, we want to estimate $\|\sum_{|v|=m} \mu(v)S^v\|$ $(\alpha_v = {}^t\lambda\mu(v)\gamma)$.
In fact, by taking $\mu : [d]^* \to M_n(\mathbb{C})$ so that n is minimal, ${}^\exists\{u_k\}_{k=1}^K$, $\{w_l\}_{l=1}^L \subset [d]^*$ s.t. we have for any $v \in [d]^*$ (Fliess, Schützenberger),

$$\mu(v)_{ij} = \sum_{k,l} c_{ij}^{kl} \alpha_{(u_k v w_l)}.$$

Therefore $\sum_{m=0}^\infty \sum_{|v|=m} \mu(v)S^v$ converges in operator norm.

The end of the proof

If $\{[r_i^*, a]\}_{i=1}^d$ are finite rank, then we get

$$\begin{aligned}
\hat{a} = a\Omega = \sum_{v \in [d]^*} \alpha_v \hat{U}_v \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} ({}^t e_1 \otimes I_2)^t \lambda \left[\sum_{m=0}^{\infty} \sum_{|v|=m} \mu(v) S^v \right] \gamma(e \otimes I_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Omega \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} ({}^t e_1 \otimes I_2)^t \lambda [1 - \sum_{i=1}^{d} \mu(i) S_i]^{-1} \gamma(e \otimes I_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Omega.
\end{aligned}$$

Since Ω is separating,

$$a = \begin{pmatrix} 1 & 0 \end{pmatrix} ({}^t e_1 \otimes I_2)^t \lambda [1 - \sum_{i=1}^{d} \mu(i) S_i]^{-1} \gamma(e \otimes I_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in C_{\mathrm{rat}}(\mathbf{s}).$$

Extension to affiliated operators

- $\widetilde{L^\infty(\mathbf{s})}$: $*$-algebra of closed operators affiliated with $L^\infty(\mathbf{s})$.
- $D(\mathbf{s})$: the division closure of $\mathbb{C}\langle \mathbf{s} \rangle$ in $\widetilde{L^\infty(\mathbf{s})}$.

Theorem (M. 2021 cf. PA. Linnel, 2000)

Let $u = f^{-1}a = bg^{-1} \in \widetilde{L^\infty(\mathbf{s})}$ where $a, b, f, g \in L^\infty(\mathbf{s})$. Then we have

$$\{fr_i^* b - ar_i^* g\}_{i=1}^d \text{ are finite rank operators} \Leftrightarrow u \in D(\mathbf{s}).$$

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Remark

We can see $fr_i^* b - ar_i^* g$ as $f[r_i^*, u]g$. Thus this can be seen an extension of the main theorem.

Future perspective

1. To extend this result to more general operators (with a dual system?). e.g. q-deformed Fock space, free tuple of real random variables with special orthogonal polynomials, other groups,...
2. When we see $\{[r_i^*, a]\}_{i=1}^d$ as the free version of Hankel operator w.r.t. $a \in L^\infty(\mathbf{s})$, are there other nice analogues of functional analysis for (classical) Hankel operators?

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Thank you for your attention!!