Riesz transforms on compact quantum groups and strong solidity

Martijn Caspers – TU Delft
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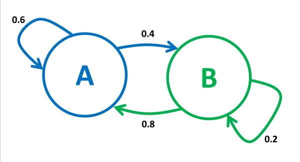
Quantum Markov semi-groups

Markov process: Probabilistic process in which the subsequent state solely depends on the current state, and does not remember anything from the past.



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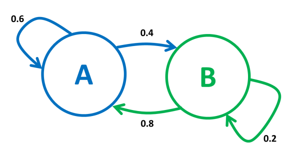

Quantum probability:

- Probability space ⇒ von Neumann algebra with a trace.
- State ⇒ density operators (positive, trace 1).
- Markov maps ⇒ trace preserving normal unital completely positive (ucp) maps (quantum channels).

Basic definitions

Von Neumann algebra: $*$-subalgebra M of $B(H)$ that is closed in the strong topology.

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Examples: $VN(\Gamma) = \overline{\{\lambda_w \mid w \in \Gamma\}}^{\text{strong}}$ with Γ group, $\lambda_w \in B(\ell_2(\Gamma))$ with $\lambda_w \delta_s = \delta_{ws}$.

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$L_2(M) = L_2(M, \tau)$ GNS-space with respect to τ. Completion of M with respect to

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$\Phi : M \to M$ is called completely positive if

$$\mathrm{id}_n \otimes \Phi : M_n(\mathbb{C}) \otimes M \to M_n(\mathbb{C}) \otimes M$$

is positive for all n, i.e. maps positive elements to positive elements.

Quantum Markov semi-groups

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A quantum Markov semi-group $(\Phi_t)_{t \geq 0}$ is a semi-group of normal unital completely positive (ucp) maps on a von Neumann algebra M that is point-strongly continuous.

They yield L_2-maps by Kadison-Schwarz,

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Unbounded generator $\Delta :\subseteq L_2(M) \to L_2(M)$ such that,

$$\Phi_t^{(2)}(x) = \exp(-t\Delta).$$

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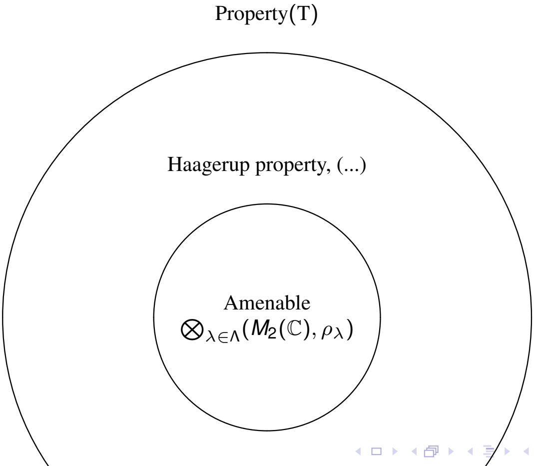
Example: $VN(\mathbb{F}_n)$ with $\Phi_t(\lambda_w) \exp(-t|w|)\lambda_w$ [Haagerup '78/79].

Approximation properties

Some approximation/rigidity properties of von Neumann algebras:



Applications: amenability

Definition: A von Neumann algebra M is amenable if there is a net of normal finite rank unital completely positive maps $\Phi_i : M \to M$ such $\forall x \in M$ we have $\Phi_i(x) \to x$ strongly.

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Theorem (Cipriani-Sauvageot '17, see also C 20)

M is amenable iff $\exists$ a quantum Markov semi-group with generator Δ with complete set of eigenvalues $\Delta_k, k \in \mathbb{N}$ (multiplicity allowed) such that

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Example: Let $\alpha \in (0, \infty)$. Then

$$\lambda_w \mapsto e^{-t|w|^\alpha} \lambda_w$$

is a QMS on $VN(\mathbb{F}_n)$ if and only if $0 < \alpha \leq 1$ [see also Bozejko].

Applications: Haagerup property

Definition: A finite von Neumann algebra M has Haagerup property if there is a net of normal trace preserving unital completely positive maps $\Phi_i : M \to M$ such that $\Phi_i^{(2)}$ is compact and $\forall \xi \in L_2(M)$ we have $\|\Phi_i^{(2)}\xi - \xi\| \to 0$.

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Example: $VN(\mathbb{F}_n)$ has Haagerup property, since

$$\lambda_w \mapsto e^{-t|w|}\lambda_w$$

is a QMS on $VN(\mathbb{F}_n)$ [Haagerup ' 78/79].

Crash course strong solidity

M is called strong solidity [Ozawa-Popa '07] if for any diffuse amenable von Neumann subalgebra $B \subseteq M$ the normalizing algebra

$$\{u \in M \text{ unitary } \mid uBu^* = B\}''$$

is again amenable.

Remark: In particular, strong solidity + non-amenability implies:

$$M \not\simeq L_\infty(X) \rtimes \Lambda, \qquad M \not\simeq M_1 \otimes M_2.$$

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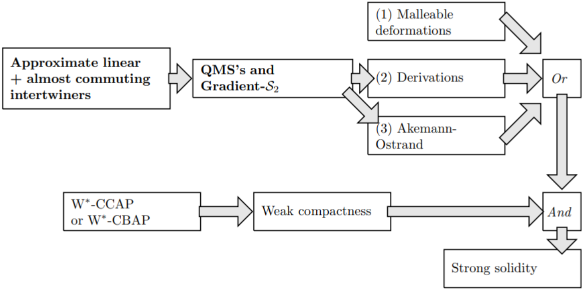

References: Ozawa, Ioana, Popa, Vaes, Isono, Peterson, Chifan, Sinclair, Udrea, ...

Gradients of quantum Markov semi-group

$\pm$ Theorem (Cipriani-Sauvageot): $\exists$ is a derivation ∇ that is the square root of Δ

Δ generator of a quantum Markov semi-group on M with some extra technical conditions omitted. There exists

- A subspace $\mathrm{Dom}(\nabla) \subseteq \mathrm{Dom}(\Delta) \subseteq L_2(M)$ that is moreover a $*$-algebra,
- An M-M-bimodule $\mathcal{H}_\nabla$,
- A closable derivation $\nabla : \mathrm{Dom}(\nabla) \to \mathcal{H}_\nabla$,

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Definition gradient bimodule $\mathcal{H}_\nabla$. Assume for simplicity a dense $*$-algebra $\mathrm{Dom}(\nabla) \subseteq \mathrm{Dom}(\Delta)$. Consider inner products on $\mathrm{Dom}(\nabla) \otimes \mathrm{Dom}(\nabla)$ by

$$\langle a \otimes b, c \otimes d \rangle = \langle \Gamma(a, c) b, d \rangle_\tau,$$

with gradient

$$\Gamma(a, c) = \frac{1}{2}(c^* \Delta(a) + \Delta(c)^* a - \Delta(c^* a)).$$

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$\mathcal{H}_\nabla$ is the completion of $\mathrm{Dom}(\nabla) \otimes \mathrm{Dom}(\nabla)$ modulo its degenerate part. Set,

$$x \cdot (a \otimes b) = xa \otimes b - x \otimes ab, \qquad (a \otimes b) \cdot x = a \otimes bx,$$
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Leibniz rule: $\nabla(xy) = x\nabla(y) + \nabla(x)y$. Root: $\nabla^* \overline{\nabla} = \Delta$.

Non-commutative Riesz transforms

We now define

$$\nabla\Delta^{-\frac{1}{2}} : L_2(M) \to \mathcal{H}_\nabla$$

called the Riesz transform.

The Riesz transform is isometric:

$$\begin{aligned}\langle \nabla\Delta^{-\frac{1}{2}}(x), \nabla\Delta^{-\frac{1}{2}}(x)\rangle_{H_\nabla} &= \langle \nabla^*\nabla\Delta^{-\frac{1}{2}}(x), \Delta^{-\frac{1}{2}}(x)\rangle_{L_2(M)} \\ &= \langle \Delta\Delta^{-\frac{1}{2}}(x), \Delta^{-\frac{1}{2}}(x)\rangle_{L_2(M)} \\ &= \langle x, x\rangle_{L_2(M)},\end{aligned}$$

(for x in a suitable dense domain).

Crash course Deformation-Rigidity theory

Definition: Akemann-Ostrand

A (finite) von Neumann algebra M has the Akemann-Ostrand property if there exists a dense unital C*-subalgebra $A \subseteq M$ such that

1. A is locally reflexive.
2. There exists a ucp map

$$\theta : A \otimes_{\min} A^{\mathrm{op}} \to B(L_2(M))$$

such that $\theta(a \otimes b^{\mathrm{op}}) - ab^{\mathrm{op}}$ is compact for all $a, b \in A$.

Crash course Deformation-Rigidity theory

Suppose that $A \subseteq M$ is a locally reflexive C*-subalgebra.

Proposition (C, Isono, Wasilewski)

Suppose that

1. H_∇ is weakly contained in $L_2(M) \otimes L_2(M)$.
2. $(a \otimes b^{\mathrm{op}}) \circ \nabla\Delta^{-\frac{1}{2}} = \nabla\Delta^{-\frac{1}{2}} \circ \pi_l(a)\pi_r(b^{\mathrm{op}})$ is compact $\forall a, b \in A$.

Then M satisfies the Akemann-Ostrand property.

Proof. $\theta(a \otimes b^{\mathrm{op}}) := (\nabla\Delta^{-\frac{1}{2}})^*(a \otimes b^{\mathrm{op}})\, \nabla\Delta^{-\frac{1}{2}}$ will do.

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Theorem (C-Isono-Wasilewski): Condition 2 is in most cases easy to check.

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Crash course Deformation-Rigidity theory

Suppose that $A \subseteq M$ is a locally reflexive C^*-subalgebra.

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Then M satisfies the Akemann-Ostrand property.

If for all $a, b \in \mathcal{A} \subseteq M$, $t \geq 0$ the following map

$$\Psi_t^{a,b} : x \mapsto \Phi_t(\Delta(axb) + a\Delta(x)b - a\Delta(xb) - \Delta(ax)b)$$

extends to a bounded map $L_2(M) \to L_2(M)$ that is moreover Hilbert-Schmidt for $t > 0$, then we say that Φ is gradient-$\mathcal{S}_2$.

Crash course Deformation-Rigidity theory

Suppose that $A \subseteq M$ is a locally reflexive C*-subalgebra.

Proposition (C, Isono, Wasilewski)

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1. $H_\triangledown$ is weakly contained in $L_2(M) \otimes L_2(M)$.
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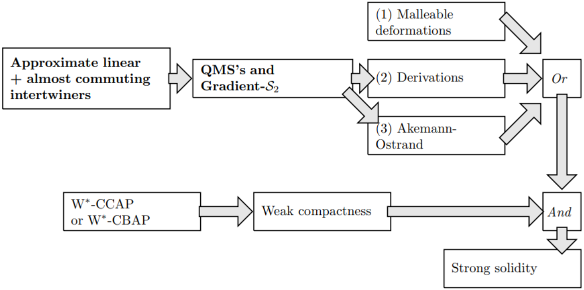
Proposition (C)

Gradient-$\mathcal{S}_2$ implies that $H_\triangledown$ is weakly contained in $L_2(M) \otimes L_2(M)$ (Condition 1).

Crash course strong solidity



Gaussian algebras

Fix $H = \mathbb{C}^n$ finite dimensional Hilbert space $\Rightarrow$ Set Fock space:

$$F = \mathbb{C}\Omega \oplus H \oplus (H \otimes H) \oplus (H \otimes H \otimes H) \oplus (H \otimes H \otimes H \otimes H) \oplus \ldots$$

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Consider creation and annihilation operators:

$$\begin{aligned} a^*(\xi) :& \eta_1 \otimes \dots \otimes \eta_n = \xi \otimes \eta_1 \otimes \dots \otimes \eta_n, \\ a(\xi) :& \eta_1 \otimes \dots \otimes \eta_n = \langle \xi, \eta_1 \rangle \eta_2 \otimes \dots \otimes \eta_n. \end{aligned}$$

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Voiculescu's free Gaussian algebra: $\Gamma(H) = \{a(\xi) + a^*(\xi) \mid \xi \in H\}''$.

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Remark: Can q-symmetrize the inner product $\Rightarrow$ q-Gaussian algebras $q \in [-1, 1]$ (Bozejko-Speicher, 1993).

- $q = 1$ bosonic.
- $q = -1$ fermionic, harmonic oscillator.
- $q = 0$ free, c.f. above.

[Bozejko-(Kümmerer)-Speicher].

Quantum Markov semi-groups

Example (continued): $\xi_1 \otimes \ldots \otimes \xi_n \in F$ in the Fock space. Then $\exists$ $W(\xi_1 \otimes \ldots \xi_n) \in \Gamma(H)$ such that

$$W(\xi_1 \otimes \ldots \otimes \xi_n)\Omega = \xi_1 \otimes \ldots \otimes \xi_n.$$

Note: this is quantization.

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The Fock space semi-group

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lifts to the algebra level

$$\Phi_t : \Gamma(H) \to \Gamma(H) : W(\xi_1 \otimes \ldots \otimes \xi_n) \mapsto e^{-tn} W(\xi_1 \otimes \ldots \otimes \xi_n).$$

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$(\Phi_t)_{t \geq 0}$ is a quantum Markov semi-group (Ornstein-Uhlenbeck semi-group).

Quantum Markov semi-groups

Theorem (C-Isono-Wasilewski)

$(\Phi_t)_{t\geq 0}$ is immediately gradient-$\mathcal{S}_2$ if

$$|q| \leq \dim(H)^{-1/2},$$

and consequently $\Gamma_q(H)$ has the Akemann-Ostrand property.

Theorem (Shlyakhtenko)

$\Gamma_q(H)$ has the Akemann-Ostrand property for $|q| < \sqrt{2} - 1$ and H finite dimensional.

Theorem (Avsec)

$\Gamma_q(H)$ is strongly solid for all $q \in (-1, 1)$ and H finite dimensional.

Open questions:

- Strong solidity with H infinite dimensional.
- Akemann-Ostrand property beyond the range $|q| \leq \max(\sqrt{2} - 1, \dim(H)^{1/2})$.

Quantum groups

A compact quantum group is a pair $\mathbb{G} := (A, \Delta_A)$ with A a unital C*-algebra and $\Delta_A : A \to A \otimes A$ a comultiplication such that

$$(\Delta_A \otimes \mathrm{id})\Delta_A = (\mathrm{id} \otimes \Delta_A)\Delta_A,$$

and such that

$$\Delta_A(A)(A \otimes 1) \quad \text{and} \quad \Delta_A(A)(1 \otimes A),$$

are dense in $A \otimes A$.

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Free orthogonal quantum group O_N^+ is generated by a matrix $u = (u_{ij})_{ij}$ with the relations that u is unitary and $\overline{u} = u$. Compultiplication:

$$\Delta_{O_N^+}(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

Quantum groups

Theorem (C)

Let $\mathbb{G} = (A, \Delta_A)$ be a quantum group with tracial Haar state τ that can be obtained from O_N^+ (equivalently $SU_q(2)$) through (repeated) applications of:

- Taking free products;
- Taking any monoidally equivalent compact quantum group;
- Taking a dual quantum subgroup;
- Taking a free wreath product with S_N^+ [Lemeux-Tarrago];
- Taking a tensor product with a finite (quantum) group.

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- All non-crossing (non-colored) partition/easy quantum groups clasified by Banica-Speicher and Weber.
- Hyperoctahedral series of quantum groups.