A metric caracterisation of freeness

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Symmetric random walks on groups

G discrete group, $g_1, \ldots, g_N \in G$,
$\mu = \frac{1}{2N} \sum_{i=1}^{N} \delta_{g_i} + \delta_{g_i^{-1}}$.
Consider the random walk on G with transition probability μ *i.e.*

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$$f \longmapsto \frac{1}{2N} \sum_{i=1}^N f(g_i \cdot) + f(g_i^{-1} \cdot).$$

Theorem (Kesten '58)

The following are equivalent:

1. $g_1, \dots, g_n$ are free,
2. $\|M_{g_1,\dots,g_N}\| = \frac{\sqrt{2N-1}}{N}$.

$S = \{g_1, \dots, g_N, g_1^{-1}, \dots, g_N^{-1}\}$.

- $(L(G), \tau_G)$ noncommutative probability space. $\lambda : G \to L(G)$.

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- We have:

$$\begin{aligned}
\|M_{g_1,\dots,g_N}\| &= \lim_{n\to\infty} \tau_G(M_{g_1,\dots,g_N}^{2n})^{\frac{1}{2n}} \\
&= \frac{1}{2N} \lim_{n\to\infty} \left(\sum_{h_1,\dots,h_{2n}\in S} \tau_G(h_1 \dots h_{2n}) \right)^{\frac{1}{2n}} \\
&= \lim_{n\to\infty} \mathbb{P}[S_{2n} = e]^{\frac{1}{2n}}.
\end{aligned}$$

Further remarks

$s_1, \ldots, s_N$ free generators of $\mathbb{F}_N$

- Consider $\pi : \mathbb{F}_N \to G$ by $\pi(s_i) = g_i$, μ' the uniform measure on the s_i and their inverses and (S'_n) the associated random walk on $\mathbb{F}_N$ such that $\pi(S'_n) = S_n$. Then:

$$\mathbb{P}[S_{2n} = e] = \mathbb{P}[S'_{2n} \in \ker\pi].$$

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- Link with amenability:

Theorem (Kesten)

The group generated by the g_i's is amenable if and only if $\|M_{g_1,\ldots,g_n}\| = 1$.

Main theorem

$(\mathcal{M}, \tau_{\mathcal{M}})$ a noncommutative probability space,
$\mathcal{M}$ a von Neumann algebra, $\tau_{\mathcal{M}}$ a normal faithful tracial state on $\mathcal{M}$
$u_1, \ldots, u_N \in \mathcal{U}(\mathcal{M})$.

$$M_{u_1,\ldots,u_N} = \sum_{i=1}^{N} u_i \otimes \overline{u_i} + u_i^* \otimes \overline{u_i^*} \in \mathcal{M}\overline{\otimes}\mathcal{M}.$$

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Theorem (Collins, C. '21)

The following are equivalent:

1. the u_i are free Haar unitaries with respect to $\tau_{\mathcal{M}}$,
2. $\|M_{u_1,\ldots,u_N}\| = 2\sqrt{2N-1}$.

Freeness VS Amenability

A caracterisation of amenability

Assume that $\mathcal{M}$ is a factor. Then following are equivalent:

1. $\mathcal{M}$ is amenable
2. for every $N \geq 0$ and every family of unitaries $u_1, \ldots, u_N$,

$$\left\| \sum_{i=1}^{N} u_i \otimes \overline{u_i} + u_i^* \otimes \overline{u_i^*} \right\|_{\mathcal{M} \otimes^{\min} \mathcal{M}} = 2N.$$

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- Open problem: does there exist N and δ such that for any factor $\mathcal{M}$ which is not amenable, there exists $u_1, \ldots, u_N$ in $\mathcal{U}(\mathcal{M})$ such that

$$\| M_{u_1, \ldots, u_N} \| < 2N - \delta.$$

The von Neumann conjecture

Let G be a group.

Conjecture (disproved)

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If $\mathcal{M}$ is a non-amenable type II_1 factor, it contains $L(\mathbb{F}_2)$.

- Gaboriau and Lyons solved a related question and obtained the following

Theorem (Gaboriau, Lyons)

Let Γ be a countable discrete non-amenable group and H be an infinite group then the factor $L(H \wr \Gamma)$ contains $L(\mathbb{F}_2)$.

$$H \wr \Gamma = \left(\bigoplus_{\gamma\in\Gamma} H_\gamma\right) \rtimes \Gamma.$$

Non-tracial case

$\mathcal{H}$ a Hilbert space, $g_1, \ldots, g_N \in \mathbb{F}_N$ free, $\lambda : \mathbb{F}_N \to L(\mathbb{F}_N)$.

Conjecture

Let $u_1, \ldots, u_N$ be unitaries in $\mathcal{B}(\mathcal{H})$. The following are equivalent:

1. for any noncommutative polynomial P,
$$\|P(u_1, \ldots, u_N)\| = \|P(\lambda(g_1), \ldots, \lambda(g_N))\|,$$
2. $\|M_{u_1,\ldots,u_N}\|_{\mathcal{B}(\mathcal{H})\otimes^{\min}\mathcal{B}(\mathcal{H})} = 2\sqrt{2N-1}$.

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- Motivated by strongly asymptotic freeness,
- the inequality $\|M_{u_1,\ldots,u_N}\| \geq 2\sqrt{2N-1}$ remains true (Pisier),
- this conjecture false! (Lehner '98).

Positive characters on $\mathbb{F}_N$

Denote by $s_1, \ldots, s_N$ free generators of $\mathbb{F}_N$.

Definition

Let G be a group and $\varphi : G \to \mathbb{C}$. We say that φ is a positive character on G if

- φ is positive definite *i.e.* for any $g_1, \ldots, g_n$ in G the matrix $(\varphi(g_i^{-1} g_j))_{1 \leq i,j \leq n}$ is positive,
- for all $g \in G$, $\varphi(g) \in \mathbb{R}_{\geq 0}$ and $\varphi(e) = 1$,
- for all $g, h \in G$, $\varphi(gh) = \varphi(hg)$.

Note that φ extends linearly to $\mathbb{C}[G]$.

- Let $u_1, \ldots, u_N$ in $\mathcal{M}$, τ_M a trace on $\mathcal{M}$,
- consider the representation π of $\mathbb{F}_N$ determined by $\pi_{u_1,\ldots,u_N}(s_i) = u_i \otimes \overline{u_i}$ for any $i \leq N$,
- then $\varphi_{u_1,\ldots,u_N} := (\tau \otimes \tau) \circ \pi$ is a positive character on $\mathbb{F}_N$.

Let $a = \sum_{i=1}^{N} s_i + s_i^{-1} \in \mathbb{C}[\mathbb{F}_N], \pi_{u_1,\dots,u_N}(a) = M_{u_1,\dots,u_N}$.

Theorem II

Let φ be a positive character on $\mathbb{F}_N$ different from δ_e,

$$\lim_{n\to\infty} \varphi(a^{2n})^{\frac{1}{2n}} > 2\sqrt{2N-1}.$$

Theorem I

The following are equivalent:

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- the u_i's are not free Haar unitaries $\Leftrightarrow \varphi_{u_1,\dots,u_N} = \delta_e$,
- furthermore:

$$\|M_u\| = \lim_{n\to\infty} \tau_{\mathcal{M}} \otimes \tau_{\mathcal{M}}(M_{u_1,\dots,u_N}^{2n})^{\frac{1}{2n}} = \lim_{n\to\infty} \varphi(a^{2n})^{\frac{1}{2n}}.$$

Lower bound for φ

Let φ be a positive character and $h_0 \in \mathbb{F}_N \setminus \{e\}$ with $\varphi(h_0) > 0$.

A remark

It suffices to show that there exists $n \in \mathbb{N}$ such that:

$$\varphi(a^{2n}) > 2^{2n}(2N-1)^n.$$

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A consequence of the fact that φ is positive definite

Let $h_1, \ldots, h_k \in \mathbb{F}_N$ tels que $\varphi(h_i) \geq \alpha > 0$ for all i. Then:

$$\sum_{i,j \leq k} \varphi(h_i^{-1} h_k) \geq n^2 \alpha^2.$$

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Two ways to generate elements of positive trace: **conjugate** or **multiply**.

- denote by $\mathcal{C}_0$ the conjugation class of h_0,
- we are going to look at the conjugates of elements of $\mathcal{C}_0^2$.

Counting paths

- Let $\mathbb{P}_n$ be the set of paths of length n starting from e in $\mathbb{F}_N$,
- for $p \in \mathbb{P}_n$ denote by $t(p)$ the element where the path stops,
- denote by $N_{n,k}$ the number of paths of length n stopping at a given element of length k,
- note that

$$\varphi(a^{2n}) = \sum_{p \in \mathbb{P}_{2n}} \varphi(t(p)) = \sum_{k=0}^{\infty} N_{2n,2k} \sum_{|g|=2k} \varphi(g).$$

Catalan's triangle

$$N_{2n,2k} \geq \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) (2N-1)^{n-k}.$$