Decomposition of free cumulants

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Free cumulants and Voiculescu transform

Theorem (Voiculescu, 1986)

The generating function of a distribution μ on $\mathbb{C}[X]$

$$R_{\mu}(z) = \sum_{n=1}^{\infty} r_n z^{n-1}$$

expressed in terms of the Cauchy transform of μ by

$$R_{\mu}(z) = G_{\mu}^{-1}(z) - \frac{1}{z}$$

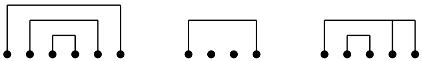
satisfies the linearization property

$$R_{\mu \boxplus \nu}(z) = R_{\mu}(z) + R_{\nu}(z),$$

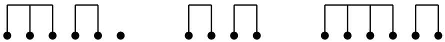
where $\mu \boxplus \nu$ is the distribution of $x + y$ for x, y free.

Noncrossing partitions and interval partitions

$\mathrm{NC}(n) \sim$ noncrossing partitions



$\mathrm{Int}(n) \sim$ interval partitions



Combinatorial definition of free cumulants

Definition (Speicher, 1994)

Free cumulants are multilinear functionals (r_n) defined by

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{NC}(n)} r_\pi[a_1, \ldots, a_n],$$

where $a_1, \ldots, a_n \in (\mathcal{A}, \varphi)$ - NPS, and

$$r_\pi[a_1, \ldots, a_n] := \prod_{V \in \pi} r_V[a_1, \ldots, a_n]$$

and

$$r_V[a_1, \ldots, a_n] := r_{|V|}(a_{i_1}, \ldots, a_{i_k})$$

when $V = \{i_1 < \cdots < i_k\}$.

Boolean cumulants

Definition (Speicher and Woroudi, 1997)

Boolean cumulants are multilinear functionals (β_n) defined by

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{Int}(n)} \beta_\pi[a_1, \ldots, a_n],$$

where

$$\beta_\pi[a_1, \ldots, a_n] := \prod_{V \in \pi} \beta_V[a_1, \ldots, a_n]$$

and

$$\beta_V[a_1, \ldots, a_n] := \beta_{|V|}(a_{i_1}, \ldots, a_{i_k})$$

when $V = \{i_1 < \cdots < i_k\}$. Again: $a_1, \ldots, a_n \in (\mathcal{A}, \varphi)$ - NPS.

Free vs boolean cumulants

Example

$$\begin{aligned}\varphi(a_1a_2a_3) &= r_3(a_1,a_2,a_3)+r_2(a_1,a_2)r_1(a_3)\\ &\quad + r_2(a_1,a_3)r_1(a_2)+r_1(a_1)r_2(a_2,a_3)\\ &\quad + r_1(a_1)r_1(a_2)r_1(a_3)\\ \varphi(a_1a_2a_3) &= \beta_3(a_1,a_2,a_3)+\beta_2(a_1,a_2)\beta_1(a_3)\\ &\quad + \beta_1(a_1)\beta_2(a_2,a_3)+\beta_1(a_1)\beta_1(a_2)\beta_1(a_3)\end{aligned}$$

Difference

There is no $\beta_2(a_1,a_3)\beta_1(a_2)$ since $\{\{1,3\},\{2\}\}\notin \mathrm{Int}(3)$.

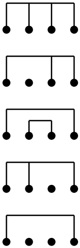
Theorem (Lehner, 2002; Belinschi and Nica, 2008)

The following combinatorial relations hold:

$$\begin{aligned} \beta_n(a_1,\ldots,a_n) &= \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} r_\pi[a_1,\ldots,a_n] \\ r_n(a_1,\ldots,a_n) &= \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1}\beta_\pi[a_1,\ldots,a_n], \end{aligned}$$

for any $a_1,\ldots,a_n \in \mathcal{A}$ and any $n\in\mathbb{N}$. Here: $\mathrm{NC}_{\mathrm{irr}}(n)$ stands for irreducible noncrossing partitions.

$\mathrm{NC}_{\mathrm{irr}}(4)$ with reversed refinement order



$A = \{\{1,2,3,4\}\}$

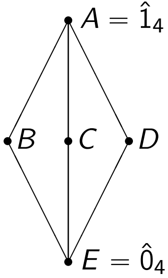
$B = \{\{1,3,4\},\{2\}\}$

$C = \{\{1,4\},\{2,3\}\}$

$D = \{\{1,2,4\},\{3\}\}$

$E = \{\{1,4\},\{2\},\{3\}\}$

$\mathrm{NC}_{\mathrm{irr}}(4)$



Examples

Using Lehner-Belinschi-Nica formula, we get

$$
\begin{aligned}
r_1(a_1) &= \beta_1(a_1), \\
r_2(a_1,a_2) &= \beta_2(a_1,a_2), \\
r_3(a_1,a_2,a_3) &= \beta_3(a_1,a_2,a_3) - \beta_2(a_1,a_3)\beta_1(a_2), \\
r_4(a_1,a_2,a_3,a_4) &= \beta_4(a_1,a_2,a_3,a_4) - \beta_3(a_1,a_2,a_4)\beta_1(a_3) \\
&\quad - \beta_3(a_1,a_3,a_4)\beta_1(a_2) - \beta_2(a_1,a_4)\beta_2(a_2,a_3) \\
&\quad + \beta_2(a_1,a_4)\beta_1(a_2)\beta_1(a_3).
\end{aligned}
$$

We are interested in the blue terms. We would like to describe them using sums of 'elementary cumulants'.

Main objective

We would like to decompose free cumulants

$$r_n(a_1, \ldots, a_n) = \sum_{w \in W} k_w(a_1, \ldots, a_n)$$

where $k_w(a_1, \ldots, a_n)$ are 'elementary cumulants' and W is a set.

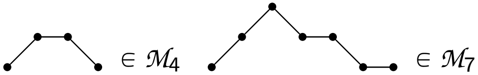
Motzkin paths

By *Motzkin paths* we understand lattice paths which

1. start from $(0,0)$ and end at $(n,0)$,
2. consist of three steps: $U=(1,1), H=(1,0)$, $D=(1,-1)$,
3. do not go below the x-axis.

Notation: Motzkin paths: $\mathcal{M}$, Motzkin paths of lenght $n-1$: $\mathcal{M}_n$.

Examples



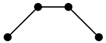
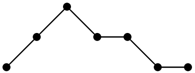
Reduced Motzkin words

Identify $\mathcal{M}$ with the set of *reduced Motzkin words*

$$w = j_1 \cdots j_n$$

where $j_1, \ldots, j_n \in \mathbb{N}$, $j_1 = j_n = 1$ and $j_k - j_{k-1} \in \{-1, 0, 1\}$.

Examples

 $\equiv$ 1221  $\equiv$ 1232211

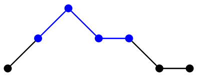
All Motzkin words

Identify Motzkin subpaths with the set $\mathcal{AM}$ of words

$$w = j_1 \cdots j_n$$

where $j_1, \ldots, j_n \geqslant j$, $j_1 = j_n = j$ and $j_k - j_{k-1} \in \{-1, 0, 1\}$. The number j is called the *height* of w, $h(w) = j$.

Example



$2322 \in \mathcal{AM}_4$

Partitions $\mathcal{M}(w)$

Definition

Monotone noncrossing partitions adapted to $w = j_1 \cdots j_n \in \mathscr{M}$ are pairs

$$\pi = (\pi_0, w) \equiv \{(V_1, v_1), \ldots, (V_k, v_k)\}$$

where

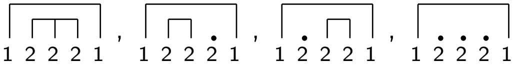
1. $\pi_0 := \{V_1, \ldots, V_k\} \in \mathrm{NC}(n)$,
2. each v_i is a constant subword of w,
3. nesting is monotone:

$$d(V_i) = h(v_i)$$

4. Notation: $\mathcal{M}(w)$ and $\mathcal{M}_{\mathrm{irr}}(w)$.

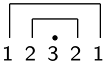
Examples

$\mathcal{M}(w)$ for $w = 12^31$

$\mathcal{M}(w) = \{$  $\}$

$\hat{1}_w$ $\hat{0}_w$

$\mathcal{M}(w)$ for $w = 12321$

$\mathcal{M}(w) = \{$  $\}$

$\hat{1}_w = \hat{0}_w$

Extending $(\mathcal{A}, \varphi)$

Definition

Let $(\mathcal{A}, \varphi)$ be a NPS and let $p^2 = p$. By the *p-extension* of $(\mathcal{A}, \varphi)$ we understand the pair $(\widetilde{\mathcal{A}}, \widetilde{\varphi})$, where

$$\widetilde{\mathcal{A}} = \mathcal{A} \star \mathbb{C}[p]$$

and $\widetilde{\varphi}$ is the linear extension of

$$\widetilde{\varphi}(p^{\alpha} a_1 p a_2 p \cdots p a_m p^{\beta}) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_m),$$

where $\alpha, \beta \in \{0, 1\}$ and $a_1, \ldots, a_m \in \mathcal{A}$.

Definition

Let $(\widetilde{\mathcal{A}}_1, \widetilde{\varphi}_1)$ be a *p*-extension of $(\mathcal{A}_1, \varphi_1)$ and let $(\widetilde{\mathcal{A}}_2, \widetilde{\varphi}_2)$ be a *q*-extension of $(\mathcal{A}_2, \varphi_2)$. Let

$$\mathcal{A} := \widetilde{\mathcal{A}}_1^{\otimes\infty} \otimes \widetilde{\mathcal{A}}_2^{\otimes\infty} \quad \text{and} \quad \Phi := \widetilde{\varphi}_1^{\otimes\infty} \otimes \widetilde{\varphi}_2^{\otimes\infty}$$

and consider the noncommutative probability space $(\mathcal{A}, \Phi)$.

Orthogonal replicas

Definition

Elements of $(\mathcal{A}, \Phi)$ of the form

$$x(j) = (\cdots \otimes \underbrace{x}_{\text{site } j} \otimes \cdots) \otimes (\cdots \otimes \underbrace{q^{\perp}}_{\text{site } j-1} \otimes q \otimes \cdots),$$

$$y(j) = (\cdots \otimes \underbrace{p^{\perp}}_{\text{site } j-1} \otimes p \otimes \cdots) \otimes (\cdots \otimes \underbrace{y}_{\text{site } j} \otimes \cdots)$$

will be called *orthogonal replicas* of $x \in \mathcal{A}_1$ and $y \in \mathcal{A}_2$ of *color* $j \in \mathbb{N}$ and *labels* 1 and 2.

Explicitly

$$
\begin{aligned}
x(1) &= (x \otimes 1_1 \otimes 1_1 \otimes 1_1 \cdots) \otimes (q \otimes q \otimes q \cdots) \\
x(2) &= (1_1 \otimes x \otimes 1_1 \otimes 1_1 \cdots) \otimes (q^{\perp} \otimes q \otimes q \cdots) \\
x(3) &= (1_1 \otimes 1_1 \otimes x \otimes 1_1 \cdots) \otimes (1_2 \otimes q^{\perp} \otimes q \cdots) \\
&\cdots
\end{aligned}
$$

and

$$
\begin{aligned}
y(1) &= (p \otimes p \otimes p \cdots) \otimes (y \otimes 1_2 \otimes 1_2 \otimes 1_2 \cdots) \\
y(2) &= (p^{\perp} \otimes p \otimes p \cdots) \otimes (1_2 \otimes y \otimes 1_2 \otimes 1_2 \cdots) \\
y(3) &= (1_1 \otimes p^{\perp} \otimes p \cdots) \otimes (1_2 \otimes 1_2 \otimes y \otimes 1_2 \cdots) \\
&\cdots
\end{aligned}
$$

Free independence from replicas

Theorem (1998)

The mixed moments of families

$$\{x_k = \sum_{j=1}^{\infty} x_k(j) : k \in K_1\} \quad \text{and} \quad \{y_k = \sum_{j=1}^{\infty} y_k(j) : k \in K_2\}$$

are free with respect to Φ.

Theorem (2020)

Let $a_k \in \mathcal{A}_{i_k}$, where $k = 1, \ldots, n$. Then we have

1. orthogonality: $a_k(j_k) \perp a_{k+1}(j_m)$ if $i_k = i_{k+1}$ and $j_k \neq j_{k+1}$,
2. freeness condition:
$$\Phi\left(a_1(j_1) \cdots a_n(j_n)\right) = 0$$
if each $a_k \in \mathrm{Ker}(\varphi_{i_k})$, $i_1 \neq \cdots \neq i_n$, and $j_1, \ldots, j_n \in \mathbb{N}$
3. reduction condition:
$$\Phi\left(a_1(j_1) \cdots 1_{i_r}(j_r) \cdots a_n(j_n)\right) = \Phi\left(a_1(j_1) \cdots \check{1}_{i_r}(j_r) \cdots a_n(j_n)\right)$$
if $a_k \in \mathrm{Ker}(\varphi_{i_k})$ for $k = 1, \ldots r-1$, $i_1 \neq \cdots \neq i_n$ and $(j_1, \ldots, j_r) = (1, \ldots, r)$, and otherwise $LHS = 0$.

Decomposition of free moments

Theorem (2020)

Let $(\mathcal{A}_1, \varphi_1)$, $(\mathcal{A}_2, \varphi_2)$ be NPS and let $a_1, \ldots, a_n \in \mathcal{A}_1 \cup \mathcal{A}_2$. Then $\varphi : \mathcal{A}_1 * \mathcal{A}_2 \to \mathbb{C}$ given by

$$\varphi(a_1 \cdots a_n) = \sum_{w=j_1\cdots j_n \in \mathcal{M}} \Phi(a_1(j_1) \cdots a_n(j_n))$$

where $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ and $i_1 \neq \cdots \neq i_n$, satisfies

$$\varphi = (\varphi_1 \star \varphi_2) \circ \tau$$

where τ is the unit identification map.

Boolean cumulants of replicas

Theorem (2020)

The Boolean cumulants of the orthogonal replicas with respect to Φ take the form

$$\beta_n(a_1(j_1), \ldots, a_n(j_n)) = \sum_{\substack{\pi \in \mathcal{M}_{\mathrm{irr}}(w) \\ \text{alternating labels}}} \beta_\pi[a_1, \ldots, a_n]$$

where $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ and $w = j_1 \cdots j_n \in \mathscr{M}$. By abuse of notation, we use β for Boolean cumulants associated with Φ.

Remark

This is a refinement of the formula for Boolean cumulants of free random variables of Fevrier, Mastnak, Nica, Szpojankowski (2019).

Free independence in terms of Boolean cumulants

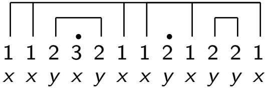
Theorem (Fevrier, Mastnak, Nica, Szpojankowski, 2019)

The Boolean cumulants of free random variables can be expressed in terms of marginal Boolean cumulants:

$$\beta_n(a_1, \cdots, a_n) = \sum_{\substack{\pi \in \mathrm{NC}_{\mathrm{irr}}(n) \\ \text{alternating labels}}} \beta_\pi[a_1, \ldots, a_n]$$

where $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$.

Alternating labels



New devices

Observation

To decompose free cumulants, we need

1. larger lattices: $\mathcal{NC}(w)$
2. conditional expectation: E

Partitions $\mathcal{NC}(w)$

Definition

Noncrossing partitions adapted to $w = j_1 \cdots j_n \in \mathcal{M}$ are pairs

$$\pi = (\pi_0, w) \equiv \{(V_1, v_1), \ldots, (V_k, v_k)\}$$

where

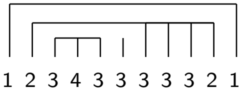
1. $\pi_0 := \{V_1, \ldots, V_k\} \in \mathrm{NC}(n)$,
2. $v_1, \ldots, v_k$ are Motzkin subwords of w,
3. nesting is appropriate:

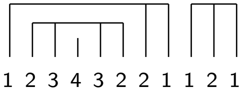
$$d(block) \leqslant h(block) \geqslant h(bridge)$$

4. neighboring blocks of same depth have equal heights.
5. Notation: $\mathcal{NC}(w)$ and $\mathcal{NC}_{\mathrm{irr}}(w)$.

Examples

Examples of $\pi \in \mathcal{NC}(w)$





It holds that

$$d(block) \leqslant h(block) \geqslant h(bridge)$$

For instance, in the first partition:

$$V_3 = \{3, 4, 5\}, \; v_3 = 343, \; b(v_3) = 23$$

$$d(V_3) = 3, h(v_3) = 3, \; h(b(v_3)) = 2$$

Examples of $\pi \notin \mathcal{NC}(w)$

For at least one block it is not true that

$$d(block) \leqslant h(block) \geqslant h(bridge)$$

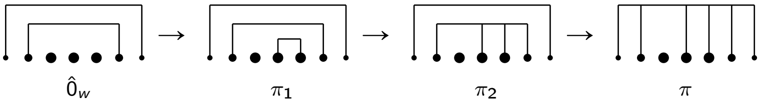
For instance:

$$V_3 = \{3,4,5\},\ v_3 = 232,\ b(v_3) = 2^2$$

$$d(V_3) = 3,\ h(v_3) = 2,\ h(b(v_3)) = 2$$

Horizontal and vertical mergings

Partitions $\pi \in \mathcal{NC}(w)$ can be constructed from the least element $\hat{0}_w$ by horizontal and vertical mergings. Let $w = 123^3 21$.



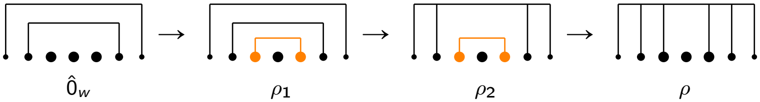
The sequence

$$\hat{0}_w \prec \pi_1 \prec \pi_2 \prec \pi$$

is a chain in $\mathcal{NC}(w)$.

Forbidden mergings

We cannot merge blocks of the same depth that are not neighbors. Let $w = 123^3 21$.



The sequence

$$\hat{0}_w \prec \rho_1 \prec \rho_2 \prec \rho$$

is not a chain in $\mathcal{NC}(w)$.

Lattices $\mathcal{NC}(w)$

Lemma 1

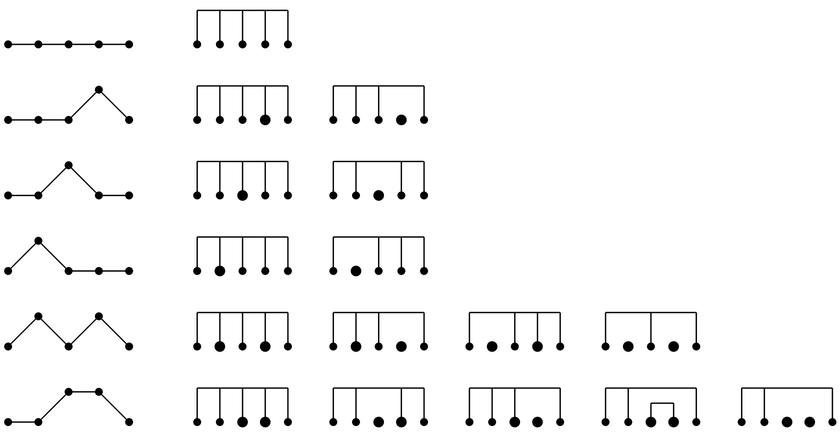
For any $w \in \mathcal{M}$, the set $\mathcal{NC}(w)$ consists of noncrossing partitions that can be obtained from $\hat{0}_w$ by a finite number of horizontal and vertical mergings.

Lemma 2

For any $w \in \mathcal{M}_n$, the set $\mathcal{NC}(w)$ is a lattice with the partial order induced from $\mathrm{NC}(n)$.

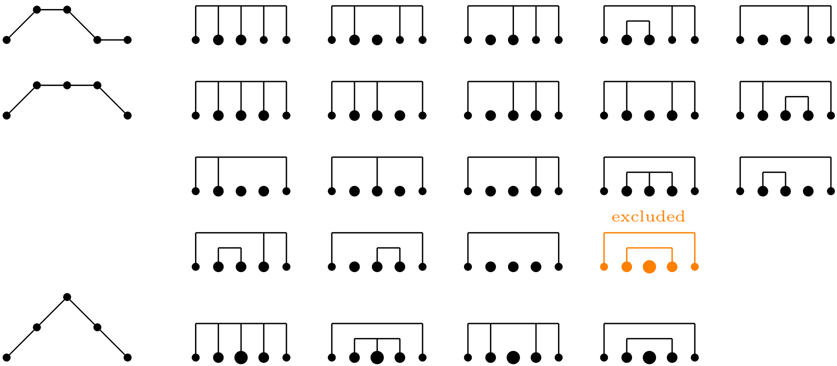
Examples of $\mathcal{NC}_{\mathrm{irr}}(w)$

w	$\mathcal{NC}_{\mathrm{irr}}(w)$

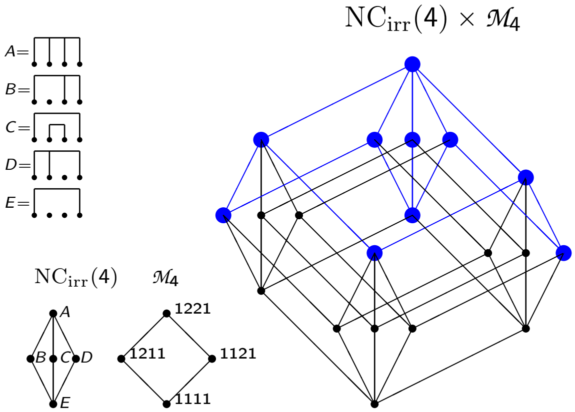


Examples of $\mathcal{NC}_{\mathrm{irr}}(w)$

w $\qquad$ $\mathcal{NC}_{\mathrm{irr}}(w)$



The Hasse diagram of $\mathrm{NC}_{\mathrm{irr}}(4) \times \mathcal{M}_4$.



Subposet $\mathcal{NC}_{\mathrm{irr}}(\mathcal{M}_4) \cong \bigsqcup_{w \in \mathcal{M}_4} \mathcal{NC}_{\mathrm{irr}}(w)$ (10 vertices).

Definition

By *interval partitions* of $w \in \mathcal{AM}$, where $h(w) = j$, adapted to w we understand the set

$$\mathcal{I}nt(w) = \{(v_1, \ldots, v_k) : w = v_1 \cdots v_k,\ v_1, \ldots, v_k \in \mathcal{AM}\},$$

where $v_1 \cdots v_k$ is a concatenation of Motzkin words of height j.

Example: $\mathcal{I}nt(w)$ for $w = 121121$ consists of 2 partitions

Definition

Let $(\mathcal{C}, \mathcal{B}, E)$ be an operator-valued NPS. By *w-Boolean cumulants* we understand the family $\{B_w : w \in \mathcal{AM}\}$ of multilinear functions

$$B_w : \mathcal{C}^n \to \mathcal{B}$$

defined by

$$E(a_1 \cdots a_n) = \sum_{\pi \in \mathcal{I}nt(w)} B_\pi[a_1, \ldots, a_n],$$

where $w = j_1 \cdots j_n \in \mathcal{AM}$, $a_1, \ldots, a_n \in \mathcal{C}$, and $B_\pi = \prod_{v \in \pi} B_v$, where B_v is the analog of β_V. In particular, $B_v := B_{\hat{1}_v}$.

Inversion formula

Proposition

If $w = j_1 \dots j_n \in \mathcal{AM}_n$, then

$$B_w(a_1, \dots, a_n) = \sum_{\pi \in \mathcal{I}nt(w)} (-1)^{|\pi|-1} E_\pi[a_1, \dots, a_n],$$

where $a_1, \dots, a_n \in \mathcal{C}$, and

$$E_\pi[a_1, \dots, a_n] = \prod_{v \in \pi} E_v[a_1, \dots, a_n],$$

where $E_v[a_1, \dots, a_n] = E(a_{i_1} \cdots a_{i_k})$ for the block $v = j_{i_1} \cdots j_{i_k}$.

Definition

Define *Motzkin cumulants*, using similar notations:

$$B_w(a_1, \ldots, a_n) = \sum_{\pi \in \mathcal{NC}_{\mathrm{irr}}(w)} K_\pi[a_1, \ldots, a_n]$$

where $a_1, \ldots, a_n \in \mathcal{C}$. In particular, $K_w = K_{\hat{1}_w}$.

Remark

This definition is a refinement of the formula

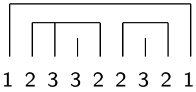
$$\beta_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} r_\pi[a_1, \ldots, a_n].$$

where $a_1, \ldots, a_n \in \mathcal{A}$.

Nesting

Remark

In operator-valued case, if π is represented by the diagram



then the corresponding partitioned cumulant is nested:

$$K_{\pi}[a_1,\ldots,a_9]=K_{11}(a_1K_{232}(a_2,a_3K_3(a_4),a_5)K_{22}(a_6K_3(a_7),a_8),a_9)$$

Example

Using the definition, we get K_w for $|w| = 4$ (10 terms):

$$\begin{aligned}
K_{1111}(a_1,\ldots,a_4) &= B_{1111}(a_1,\ldots,a_4),\\
K_{1211}(a_1,\ldots,a_4) &= B_{1211}(a_1,\ldots,a_4) - B_{111}(a_1B_2(a_2),a_3,a_4),\\
K_{1121}(a_1,\ldots,a_4) &= B_{1121}(a_1,\ldots,a_4) - B_{111}(a_1,a_2B_2(a_3),a_4),\\
K_{1221}(a_1,\ldots,a_4) &= B_{1221}(a_1,\ldots,a_4) - B_{121}(a_1B_2(a_2),a_3,a_4)\\
&\quad - B_{121}(a_1,a_2B_2(a_3),a_4)\\
&\quad - B_{11}(a_1B_{22}(a_2,a_3),a_4)\\
&\quad + B_{11}(a_1B_2(a_2)B_2(a_3),a_4).
\end{aligned}$$

Together they resemble

$$\begin{aligned}
r_4(a_1,a_2,a_3,a_4) &= \beta_4(a_1,a_2,a_3,a_4) - \beta_3(a_1,a_2,a_4)\beta_1(a_3)\\
&\quad - \beta_3(a_1,a_3,a_4)\beta_1(a_2) - \beta_2(a_1,a_4)\beta_2(a_2,a_3)\\
&\quad + \beta_2(a_1,a_4)\beta_1(a_2)\beta_1(a_3).
\end{aligned}$$

Dedication = nesting + orthogonality

Definition

Let $a_1, \ldots, a_n \in \mathcal{C}$, $w = j_1 \cdots j_n \in \mathcal{AM}$. If

1. if $v, v' \in \mathcal{B}(w)$ and $h(v) = h(v')$, then
$$B_v(a_k, \ldots, a_p B_{v'}(a_{p+1}, \ldots, a_q), a_{q+1}, \ldots, a_m) = 0,$$
2. if $v, v' \in \mathcal{B}(w)$ and $h(v) \neq h(v')$, then
$$B_v(a_k, \ldots, a_p) B_{v'}(a_{p+1}, \ldots, a_q) = 0,$$

then the family $\{B_v : v \in \mathcal{B}(w)\}$ is *dedicated* to $(a_1, \ldots, a_n)$, where $\mathcal{B}(w)$ is the set of all blocks of $\mathcal{NC}(w)$.

Inversion formula

Theorem (2023)

If $w = j_1 \cdots j_n \in \mathcal{AM}$ and the family $\{B_v : v \in \mathcal{B}(w)\}$ is dedicated to $(a_1 \ldots, a_n) \in \mathcal{C}^n$, then

$$K_w(a_1, \ldots, a_n) = \sum_{\pi \in \mathcal{NC}_{\rm irr}(w)} (-1)^{|\pi|-1} B_\pi[a_1, \ldots, a_n].$$

Remark

This formula is a refinement of the Lehner-Belinschi-Nica formula

$$r_n(a_1, \ldots, a_n) = \sum_{\pi \in {\rm NC}_{\rm irr}(n)} (-1)^{|\pi|-1} \beta_\pi[a_1, \ldots, a_n].$$

where $a_1, \ldots, a_n \in \mathcal{A}$.

Projections

Definition

Return to $(\mathcal{A}, \Phi)$, where we distinguish:

1. the sequence of projections $(e_n)_{n\in\mathbb{N}}$:

$$e_n := (1_1^{\otimes(n-1)} \otimes p^{\otimes\infty}) \otimes (1_2^{\otimes(n-1)} \otimes q^{\otimes\infty}),$$

2. the sequence of associated orthogonal projections:

$$p_n = e_n - e_{n-1}$$

where we set $e_0 = 0$ so that $p_1 = e_1$,

3. the unital commutative algebra

$$\mathcal{B} = \mathbb{C}[p_1, p_2, p_3, \ldots] = \mathbb{C}[e_1, e_2, e_3, \ldots].$$

Family $\mathcal{F}$

Definition

Define a more general family of projections

$$\mathcal{F} = \{e(\epsilon, \eta) : \epsilon, \eta \in \{0,1\}^{\infty}\},$$

where

$$e(\epsilon, \eta) := (p^{\epsilon_1} \otimes p^{\epsilon_2} \otimes \cdots) \otimes (q^{\eta_1} \otimes q^{\eta_2} \otimes \cdots)$$

and $\epsilon = (\epsilon_1, \epsilon_2, \ldots), \eta = (\eta_1, \eta_2, \ldots)$.

Definition

If $e = e(\epsilon, \eta)$, let

$$j(e) := \min\{j : \epsilon_j = 1 \ \text{ or } \ \eta_j = 1\}.$$

Remark

The unital subalgebra $\mathcal{A}_{\mathrm{rep}}$ generated by orthogonal replicas is spanned by elements of the form

$$eaf,$$

where $e, f \in \mathcal{F}$ and $a \in S$, where S is the set of simple tensors with no p or q on the left or on the right.

Proposition

The linear extension of $E : \mathcal{A}_{\mathrm{rep}} \to \mathcal{B}$ given by

$$E(eaf) = e_{j(e)}\Phi(a)e_{j(f)}.$$

is a conditional expectation.

Theorem (2023)

If $a_1, \ldots, a_n$ are orthogonal replicas with labels and colors encoded by $\ell = i_1 \cdots i_n$ and $w = j_1 \cdots j_n \in \mathcal{AM}$, where $h(w) = j$, then

$$B_w(a_1, \ldots, a_n) = \left(\sum_{\pi \in \mathcal{M}_{\mathrm{irr}}(w,\ell)} \beta_\pi[a_1, \ldots, a_n] \right) p_j.$$

where the B_w are w-Boolean cumulants associated with E. For identical labels and non-identical colors, $B_w(a_1, \ldots, a_n) = 0$.

Example

1. If x_1, x_2, x_3 are replicas of label 1 and colors $1, 2, 1$, then
$$E(x_1x_2x_3) = B_w(x_1, x_2, x_3) = 0,$$
by orthogonality, thus
$$K_w(x_1, x_2, x_3) = -\beta_2(x_1, x_3)\beta_1(x_2)p_1 \neq 0.$$
2. If x_1, y, x_2 are replicas of labels $1, 2, 1$ and colors $1, 2, 1$, then
$$E(x_1y_2x_3) = B_w(x_1, y_2, x_3) = \beta_2(x_1, x_3)\beta_1(y_2)p_1 \neq 0,$$
thus
$$K_w(x_1, y_2, x_3) = 0.$$

Lemma 1

Let $a_1, \ldots, a_n$ be orthogonal replicas with labels and colors encoded by $\ell = i_1 \cdots i_n$ and $w = j_1 \cdots j_n \in \mathcal{AM}$.

1. The family $\{B_v : v \in \mathcal{B}(w)\}$ is dedicated to $(a_1, \ldots, a_n)$.
2. In particular, if $h(w) = j$, then

$$B_w(a_1, \ldots, a_k p_j, a_{k+1}, \ldots, a_n) = 0.$$

3. Moreover, if $h(w) = j_k = j_{k+1} = j$ and $i_k \neq i_{k+1}$, then

$$B_w(a_1, \ldots, a_k p_{j+1}, a_{k+1}, \ldots, a_n) = 0.$$

Properties of B_w for replicas

Lemma 2

Let the same assumptions be satisfied.

1. If the labels are identical and $(j_k, j_{k+1}) = (j, j)$, then
$$B_w(a_1, \ldots, a_k p_{j+1}, a_{k+1}, \ldots, a_n) = B_w(a_1, \ldots, a_n).$$
2. If $i_k \neq i_{k+1}$ and $(j_k, j_{k+1}) \in \{(j, j+1), (j+1, j)\}$, then
$$B_w(a_1, \ldots, a_k p_{j+1}, a_{k+1}, \ldots, a_n) = B_w(a_1, \ldots, a_n).$$

Properties of K_w for replicas

Lemma 3

Under the same assumptions, with $h(w) = j$.

1. If the labels are arbitrary, then

$$K_w(a_1, \ldots, a_k p_j, a_{k+1}, \ldots, a_n) \quad = \quad 0.$$

2. If the labels are identical and $(j_k, j_{k+1}) \in \{(j,j), (j, j+1), (j+1, j)\}$, then

$$K_w(a_1, \ldots, a_k p_{j+1}, a_{k+1}, \ldots, a_n) \quad = \quad K_w(a_1, \ldots, a_n).$$

Theorem (2023)

If $a_1, \ldots, a_n$ are orthogonal replicas with identical labels and colors encoded by $w \in \mathcal{AM}$, where $h(w) = j$, then

$$K_w(a_1, \ldots, a_n) = \left(\sum_{\pi \in \mathcal{M}_{\mathrm{irr}}(w)} (-1)^{|\pi|-1} \beta_\pi [a_1, \ldots, a_n] \right) p_j.$$

If their labels are not identical, then $K_w(a_1, \ldots, a_n) = 0$.

Sketch of proof

Sketch of proof for identical labels

By Lemma 1, the family $\{B_v : v \in \mathcal{B}(w)\}$ is dedicated to $(a_1, \ldots, a_n)$. Now, by the theorem on B_w, the cumulants $B_v(a_{k_1}, \ldots, a_{k_p})$ vanish whenever the labels of $a_{k_1}, \ldots, a_{k_p}$ are identical and the colors are not. Therefore, we can conclude that the summation over $\mathcal{NC}_{\mathrm{irr}}(w)$ in the Inversion Formula reduces to that over $\mathcal{M}_{\mathrm{irr}}(w)$. Then it suffices to use Lemma 2 repeatedly to obtain

$$B_\pi[a_1, \ldots, a_n] = \beta_\pi[a_1, \ldots, a_n]p_j$$

for $\pi \in \mathcal{M}_{\mathrm{irr}}(w)$, which completes the proof in this case.
If labels are not identical, we use induction and Lemma 3. □

Summary of properties of B_w and K_w

Corollary

Let $a_1, \ldots, a_n$ be orthogonal replicas with labels $i_1, \ldots, i_n$ and colors $j_1, \ldots, j_n$ and let $w = j_1 \cdots j_n \in \mathcal{AM}$.

1. If $i_1, \ldots, i_n$ are not identical, then
$$K_w(a_1, \ldots, a_n) = 0.$$
2. If $i_1 = \cdots = i_n$ and $j_1, \ldots, j_n$ are not identical, then
$$B_w(a_1, \ldots, a_n) = 0.$$
3. If $i_1 = \cdots = i_n$ and $j_1 = \ldots = j_n$, then
$$K_w(a_1, \ldots, a_n) = B_w(a_1, \ldots, a_n).$$

Scalar-valued Motzkin cumulants

Definition

Let $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ be noncommutative probability spaces and let $\mathcal{A}^\circ$ be the vector space sum of $\mathcal{A}_1$ and $\mathcal{A}_2$. Multilinear functionals

$$k_w : \mathcal{A}^\circ \times \cdots \times \mathcal{A}^\circ \to \mathbb{C}$$

where $w = j_1 \cdots j_n \in \mathcal{M}$ defined as the multilinear extensions of

$$k_w(a_1, \ldots, a_n) = \zeta(K_w(a_1(j_1), \ldots, a_n(j_n))),$$

where $a_1, \ldots, a_n \in \mathcal{A}_1 \cup \mathcal{A}_2$ and $\zeta : \mathcal{B} \to \mathbb{C}$ is the linear extension of $\zeta(p_k) = \delta_{k,1}$, will be called *scalar-valued Motzkin cumulants.*

Theorem (2023)

Let $a_1, \ldots, a_n \in \mathcal{A}_1 \cup \mathcal{A}_2$ and let $\tau(a_1), \ldots, \tau(a_n)$ be their canonical embeddings in $\mathcal{A}_1 \star \mathcal{A}_2$. Their mixed cumulants associated with $\varphi_1 \star \varphi_2$ have the decomposition

$$r_n(\tau(a_1), \ldots, \tau(a_n)) = \sum_{w \in \mathcal{M}_n} k_w(a_1, \ldots, a_n),$$

where

$$k_w(a_1, \ldots, a_n) = \sum_{\pi \in \mathcal{M}_{\mathrm{irr}}(w)} (-1)^{|\pi|-1} \beta_\pi[a_1, \ldots, a_n]$$

if all $a_i \in \mathcal{A}_1$ or all $a_i \in \mathcal{A}_2$ and otherwise $k_w(a_1, \ldots, a_n) = 0$.

Sketch of proof

If the variables have identical labels, by theorem on K_w we have

$$k_w(a_1, \ldots, a_n) = \sum_{\pi \in \mathcal{M}_{\rm irr}(w)} (-1)^{|\pi|-1} \beta_\pi[a_1, \ldots, a_n]$$

Using the natural bijection $\eta : {\rm NC}_{\rm irr}(n) \to \bigsqcup_{w \in \mathcal{M}_n} \mathcal{M}_{\rm irr}(w)$

$$\eta(\{V_1, \ldots, V_p\}) = \{(V_1, v_1), \ldots, (V_p, v_p)\},$$

where $v_k = d_k^{|V_k|}$ and $d_k = d(V_k)$ and summing over $w \in \mathcal{M}_n$, we get $r_n(\tau(a_1), \ldots, \tau(a_n))$ for identical labels. By theorem on K_w

$$k_w(a_1, \ldots, a_n) = \zeta(K_w(a_1(j_1), \ldots, a_n(j_n))) = 0$$

if labels are not identical. $\square$

Example

First few free cumulants can be decomposed as

$$\begin{aligned} r_1(a_1) &= k_1(a_1), \\ r_2(a_1, a_2) &= k_{11}(a_1, a_2), \\ r_3(a_1, a_2, a_3) &= k_{111}(a_1, a_2, a_3) + k_{121}(a_1, a_2, a_3) \\ r_4(a_1, a_2, a_3, a_4) &= k_{1111}(a_1, a_2, a_3, a_4) + k_{1211}(a_1, a_2, a_3, a_4) \\ &+ k_{1121}(a_1, a_2, a_3, a_4) + k_{1221}(a_1, a_2, a_3, a_4) \end{aligned}$$

Example

Setting $a_1 = \ldots = a_n = a$, we obtain

$$r_n = \sum_{w \in \mathcal{M}_n} k_w = \sum_{w \in \mathcal{M}_n} \sum_{\pi \in \mathcal{M}_{\mathrm{irr}}(w)} (-1)^{|\pi|-1} \beta_\pi .$$

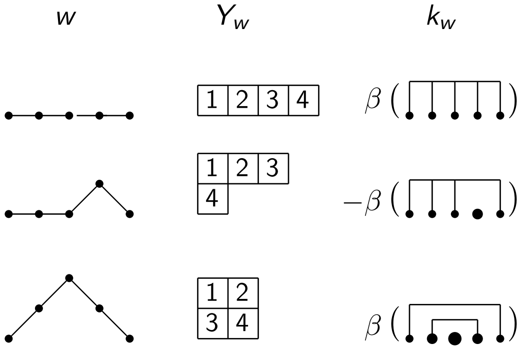
For instance,

$$\begin{aligned} r_1 &= k_1 = \beta_1, \\ r_2 &= k_{11} = \beta_2, \\ r_3 &= k_{111} + k_{121} = \beta_3 + (-\beta_2\beta_1), \\ r_4 &= k_{1111} + k_{1121} + k_{1211} + k_{1221} \\ &= \beta_4 + (-\beta_3\beta_1) + (-\beta_3\beta_1) + (-\beta_2^2 + \beta_2\beta_1^2), \end{aligned}$$

Standard Young Tableaux

Bijection $\mathcal{M}_n \cong \mathcal{T}_n^{(3)}$

For each $n \in \mathbb{N}$, there is a bijective correspondence $\mathcal{M}_n \cong \mathcal{T}_n^{(3)}$, where $\mathcal{T}_n^{(3)}$ is the set of Standard Young Tableaux with n cells and at most three rows (Eu, Fu, Hou, Hsu, 2013). Examples:



Definition

The *Motzkin homogenous part of* $\mu_1 \boxplus \mu_2$ *associated with* $w \in \mathcal{M}$ we understand the linear functional on H_n given by the linear extension of

$$(\mu_1 \boxplus_w \mu_2)(X_{k_1} \cdots X_{k_n})$$
$$= \Phi((a_{k_1,1}(j_1) + a_{k_1,2}(j_1)) \cdots (a_{k_n,1}(j_n) + a_{k_n,2}(j_n)))$$

when $|w| = n$, and we set $\mu_1 \boxplus_\varnothing \mu_2 = Id_{\mathbb{C}1}$. The family

$$\mathcal{H}(\mu_1, \mu_2) := \{\mu_1 \boxplus_w \mu_2 : w \in \mathcal{M}\}$$

is called the *family of Motzkin homogenous parts* of $\mu_1 \boxplus \mu_2$.

Proposition

It holds that

$$(\mu_1 \boxplus_w \mu_2)(X_{k_1} \cdots X_{k_n}) \;=\; \sum_{\pi \in \mathcal{M}(w)} \sum_{\ell \in \mathcal{L}_0(\pi)} \beta_\pi[a_{k_1,i_1}, \ldots, a_{k_n,i_n}]$$

for any $k_1, \ldots, k_n \in I$,where $\mathcal{L}_0(\pi)$ denotes the set of labelings ℓ to which π is monotonically adapted.

Example

Let $f = X_{k_1}X_{k_2}X_{k_3}X_{k_4}$. For simplicity, denote $x_k = a_{k,1}$, $y_k = a_{k,2}$. We obtain

$$\begin{aligned}
(\mu_1 \boxplus_{w_1} \mu_2)(f) &= (\mu_1 \uplus \mu_2)(f),\\
(\mu_1 \boxplus_{w_2} \mu_2)(f) &= \beta_3(x_{k_1}, x_{k_2}, x_{k_4})\beta_1(y_{k_3}) + \beta_1(x_{k_1})\beta_2(x_{k_2}, x_{k_4})\beta_1(y_{k_3})\\
&+ \beta_1(y_{k_1})\beta_2(x_{k_2}, x_{k_4})\beta_1(y_{k_3}) + x \rightleftarrows y,\\
(\mu_1 \boxplus_{w_3} \mu_2)(f) &= \beta_3(x_{k_1}, x_{k_3}, x_{k_4})\beta_1(y_{k_2}) + \beta_2(x_{k_1}, x_{k_3})\beta_1(y_{k_2})\beta_1(x_{k_4})\\
&+ \beta_2(x_{k_1}, x_{k_3})\beta_1(y_{k_2})\beta_1(y_{k_4}) + x \rightleftarrows y,\\
(\mu_1 \boxplus_{w_4} \mu_2)(f) &= \beta_2(x_{k_1}, x_{k_4})(\beta_2(y_{k_2}, y_{k_3}) + \beta_1(y_{k_2})\beta_1(y_{k_3})) + x \rightleftarrows y,
\end{aligned}$$

where $w_1 = 1^4$, $w_2 = 1^2 21$, $w_3 = 121^2$, $w_4 = 12^2 1$.

Convolutions in terms of K_w

Proposition

For any $k_1, \ldots, k_n \in I$ and any $w = j_1 \cdots j_n \in \mathcal{M}_n$, it holds that

$$(\mu_1 \boxplus_w \mu_2)(X_{k_1} \cdots X_{k_n})$$

$$\sum_{\pi \in \mathcal{NC}(w)} \sum_{\ell \in \mathcal{L}(\pi)} \zeta(K_\pi[a_{k_1,i_1}(j_1), \ldots, a_{k_n,i_n}(j_n)]),$$

where $\mathcal{L}(\pi)$ is the set of labelings to which π is adapted.

Example

Example

Let $w = 12^31$. In the computation of $(\mu_1 \boxplus_w \mu_2)(f)$, where $f = X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}$, the contribution associated with the partition $\pi = (\pi_0, w)$, where $\pi_0 = \{\{1,2,4,5\},\{3\}\}$, is of the form

$$\begin{aligned} & \zeta(K_{12^21}(x_{k_1}, x_{k_2}(K_2(x_{k_3}) + K_2(y_{k_3})), x_{k_4}, x_{k_5}) \\ + \quad & \zeta(K_{12^21}(y_{k_1}, y_{k_2}(K_2(x_{k_3}) + K_2(y_{k_3})), y_{k_4}, y_{k_5}) \end{aligned}$$

which, due to the fact that $K_2(x_{k_3}) + K_2(y_{k_3})$ is proportional to p_2, cannot be written in the usual product form.

Thank you

Thank you for your attention!