Cyclic independence

notions of independence $\longleftrightarrow$ graph products

classical	direct prod
free	free product
Boolean	star product
monotone	comb product

star product, comb product: preserve finiteness

1) $\sigma(\Gamma) = \sigma(A_\Gamma)$ adj. spectrum

$$\phi_\Gamma(x) = \det(x - A_\Gamma) = \prod (x - \lambda_i)$$

2) Green function

$$(A_\Gamma^n)_{ij} = \#\text{ walks from } i \to j$$

$$(A_\Gamma^n)_{1,1} = \#\text{ closed walks from root } 1$$

G.F. $\sum a_{11}^{(n)} x^n = \langle (I - x A_\Gamma)^{-1} e_1, e_1 \rangle$ Green fct

$R_\Gamma(z) = (z - A_\Gamma)^{-1}$ resolvent

$G_\Gamma(z) = \langle (z - A_\Gamma)^{-1} e_1, e_1 \rangle$ Cauchy transform

$F_\Gamma(z) = 1/G_\Gamma(z)$ recip. C.T.

$g_\Gamma(z) = Tr(R_\Gamma(z)) = \sum_{i=1}^{n} \langle (z - A_\Gamma)^{-1} e_i, e_i \rangle$

$Tr(M) = \sum \lambda_i$ $\qquad = \sum (z - \lambda_i)^{-1} = \frac{d}{dz} \log \phi_\Gamma(z)$

$Tr(f(M)) = \sum f(\lambda_i)$

Q: Relation between $G_\Gamma(z)$ and $g_\Gamma(z)$?

Schur complement

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$ assum: D invertible

$$M = \begin{bmatrix} I & BD^{-1} \\ & I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & \\ D^{-1}C & I \end{bmatrix}$$

$$S = M/_D = A - BD^{-1}C$$ Schur complement

$\rightsquigarrow$ Gelfand's nc. determinants

case $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad S = \dfrac{ad-bc}{d}$

Consequences:

1) Jacobi's id.

$$\det M = \det S \cdot \det D$$

27) Banachiewicz formula

$$M^{-1} = \begin{bmatrix} S^{-1} & * \\ * & D^{-1} + D^{-1} C S^{-1} B D^{-1} \end{bmatrix}$$

<u>in our case</u>: 1,1 entry / rest

$1 + (n-1)$

$$\begin{array}{cc} & \begin{array}{cc} 1 & n-1 \end{array} \\ A = & \begin{bmatrix} \alpha & b^* \\ b & D \end{bmatrix} \begin{array}{c} 1 \\ n-1 \end{array} \end{array} \qquad M = z - A = \begin{bmatrix} z-\alpha & -b^* \\ -c & z-D \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \left(z - \alpha - b^*(z-D)^{-1} c\right)^{-1} & \\ & \end{bmatrix} \qquad 1/G_A(z) = F_A(z)$$

$$\phi_A(z) = \det M = \left(z - \alpha - b^*(z-D)^{-1} c\right) \cdot \phi_D(z)$$

case of a graph: $v_1 =$ root

$$A_\Gamma = \begin{bmatrix} 0 & b^* \\ b & A_{\mathring{\Gamma}} \end{bmatrix}$$

$\mathring{\Gamma} = \Gamma \setminus \{v_1\}$

$$\Phi_\Gamma(z) = F_\Gamma(z)\, \Phi_{\mathring{\Gamma}}(z)$$

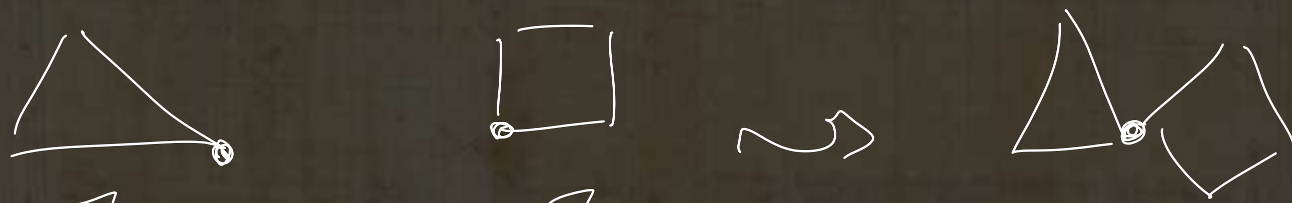
Schwenk 1970's

$$g_\Gamma(z) = \mathrm{Tr}\left((z - A_\Gamma)^{-1}\right)$$

$$= g_{\mathring{\Gamma}}(z) + G_\Gamma(z)\, F_\Gamma'(z)$$

$$= g_{\mathring{\Gamma}}(z) + \frac{d}{dz} \log F_\Gamma(z)$$

Star product



$\Gamma_1 \qquad\qquad \Gamma_2 \qquad\qquad \Gamma_1 \circledast \Gamma_2$

$$A_{\Gamma_1} = \begin{bmatrix} 0 & b_1^* \\ b_1 & \mathring{A}_1 \end{bmatrix} \qquad A_{\Gamma_2} = \begin{bmatrix} 0 & b_2^* \\ b_2 & \mathring{A}_2 \end{bmatrix}$$

$$A_{\Gamma_1 \circledast \Gamma_2} = \begin{bmatrix} 0 & b_1^* & b_2^* \\ b_1 & \mathring{A}_1 & \\ b_2 & & \mathring{A}_2 \end{bmatrix}$$

$$\longrightarrow \quad F_\Gamma(z) - z = F_{\Gamma_1}(z) - z + F_{\Gamma_2}(z) - z$$

$$\log \underline{\Phi_\Gamma} + \log G_\Gamma = \log \Phi_1 + \log G_1 + \log \Phi_2 + \log G_2$$

$$\boxed{g_\Gamma + \frac{G_\Gamma'}{G_\Gamma} = g_1 + \frac{G_1'}{G_1} + g_2 + \frac{G_2'}{G_2}}$$

embed $V(\Gamma_1 \otimes \Gamma_2) \subseteq V_1 \times V_2$

$$= \{(x_1, o_2) \mid x \in V_1\} \cup \{(o_1, x_2) \mid x_2 \in V_2\}$$

$$A \in \ell_2(V_1 \times V_2) = \ell_2(V_1) \otimes \ell_2(V_2)$$

$$A = A_1 \otimes P_2 + P_1 \otimes A_2$$

proj on o_2 proj on o_1

abstract generalization:

(H_i, ξ_i) rooted Hilbert spaces

$\|\xi_i\| = 1$ "vacuum"

$\varphi_i(A) = \langle A\xi_i, \xi_i \rangle$

$P_i = \xi_i \xi_i^*$

$H = H_1 \otimes \cdots \otimes H_N$

$\xi = \xi_1 \otimes \cdots \otimes \xi_N$

$\pi_i(A) = P_1 \otimes P_2 \cdots \otimes A \otimes P_{i+1} \cdots \otimes P_N$

are Boolean indep

$\varphi(\pi_1(A)\,\pi_2(B)) = \langle (A \otimes P_2)(P_1 \otimes B)\, \xi_1 \otimes \xi_2, \xi_1 \otimes \xi_2 \rangle$

$= \langle A P_1 \xi_1, \xi_1 \rangle \langle P_2 B \xi_2, \xi_2 \rangle$

$$= \varphi_1(A) \cdot \varphi_2(B)$$

$$\varphi\big(\pi_1(A)\, \pi_2(B)\, \pi_1(A)\big) =$$

$$\langle A P_1 A\, \xi_1, \xi_1 \rangle \cdot \langle P_2 B P_2\, \xi_2, \xi_2 \rangle$$

$$\neq \xi_1^* A \xi_1\, \xi_1^* A \xi_1 \cdot \varphi_2(B)$$

$$\varphi_1(A)^2 \cdot \varphi_2(B)$$

full trace? assum A, B <u>trace class</u>

$$\mathrm{Tr}\big(\pi_1(A)\, \pi_2(B)\big) = \mathrm{Tr}\big((A \otimes P_2)(P_1 \otimes B)\big)$$

$$= \mathrm{Tr}\,(A P_1 \otimes P_2 B)$$

$$= \mathrm{Tr}\,(A P_1) \cdot \mathrm{Tr}\,(P_2 B)$$

$$= \varphi_1(A) \cdot \varphi_2(B)$$

$$\tau\left(\pi_1(A)\,\pi_2(B)\,\pi_1(A)\right) = \tau\left(\pi_1(A^2)\,\pi_2(B)\right)$$

$$= \varphi_1(A^2)\cdot\varphi_2(B)$$

<u>Definition</u> cyclic ncps $(\mathcal{A}, \omega, \varphi)$

$\mathcal{A}$ $*$-alg

$\omega : \mathcal{D} \subseteq \mathcal{A} \longrightarrow \mathbb{C}$ (unbounded) trace

$\varphi : \mathcal{A} \to \mathbb{C}$ state

$(\mathcal{A}_i)_{i\in I} \subseteq \mathcal{A}$ are cyclic Boolean indep if

- Boolean indep wrt φ

$\forall\ a_j \in \mathcal{A}_{i_j}$, $i_j \neq i_{j+1}$

$$\varphi(a_1 \ldots a_n) = \varphi(a_1)\ldots\varphi(a_n)$$

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$$\omega(a_1 \dots a_n) = \begin{cases} \omega(a_1) & n = 1 \\ \varphi(a_1 \dots a_n) & i_1 \neq i_n \\ \varphi(a_n a_1)\varphi(a_2) \dots \varphi(a_{n-1}) & i_1 = i_n \end{cases}$$

$\to$ all calculations carry through (Schur complement)

- Problem: $g_a(z) = \omega\left((z-a)^{-1}\right)$ unbounded

$$\check{g}_a(z) = \omega\left((z-a)^{-1} - z^{-1} \cdot I\right)$$

$$h_a(z) = \tilde{g}_a(z) + \frac{d}{dz} \log z\, G_a(z) \quad \text{is additive}$$

$$h_{a+b} = h_a + h_b$$

cumulants?

better w.: $C_a(z) = \frac{1}{z} h_a\left(\frac{1}{z}\right) + z\, B_a'(z)$

$$B_a(z) = 1 - z F_a(1/z)$$

$$C_{a+b} = C_a + C_b$$

$$C_a(z) = \sum_{n=1}^{\infty} c_n(a) z^n$$

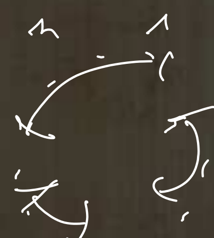
"cyclic cumulants"

$$c_n(a) = \omega(a^n) + \mathrm{poly}(\varphi(a), \dots, \varphi(a^{n-1}))$$

universal

$$\omega(a^n) = c_n(a) + \sum_{\substack{\pi \in CI_n \\ \pi \neq 1_n}} b_\pi(a)$$

↪ Boolean cumulants

$CI_n =$  interval partitions on the clock

Cyclic Boolean free product ?

<u>Given</u>: $(\mathcal{A}_1, \omega_1, \varphi_1) \circledast (\mathcal{A}_2, \omega_2, \varphi_2)$

$$\omega(a, b, a_2 \dots) = \begin{cases} \omega(a,) & n=1 \\ \vdots \text{ etc} & \end{cases}$$

<u>Problem</u>: ω is not positive