On the isomorphism class of q-Gaussian C*-algebras[1]

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[1] based on joint work with Matthijs Borst, Martijn Caspers and Mateusz Wasilewski

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The q-relations

Let $\mathcal{H}$ be a (complex and separable) Hilbert space and let $\{l(\xi) \mid \xi \in \mathcal{H}\}$ be a family of operators with $l(\xi + \eta) = l(\xi) + l(\eta)$ for all $\xi, \eta \in \mathcal{H}$.

Definition

Let $q \in \mathbb{R}$. The family satisfies the *q-relations* if for all $\xi, \eta \in \mathcal{H}$,

$$l(\xi)l(\eta)^* - q l(\xi)^* l(\eta) = \langle \xi, \eta \rangle \,\mathrm{id}.$$

The canonical commutation and anti-commutation relations are fundamental relations describing bosons and fermions respectively.

- Case $q = 1$ (*Bosonic case*): $l(\xi)l(\eta)^* - l(\xi)^* l(\eta) = \langle \xi, \eta \rangle \,\mathrm{id}$;
- Case $q = (-1)$ (*Fermionic case*): $l(\xi)l(\eta)^* + l(\xi)^* l(\eta) = \langle \xi, \eta \rangle \,\mathrm{id}$;
- Case $q = 0$ (*Cuntz relation*): $l(\xi)l(\eta)^* = \langle \xi, \eta \rangle \,\mathrm{id}$.

Such families of operators have first been considered by Frisch and Bourret in 1970.

Problem: It is not at all clear that such families of operators even exist!

Their existence was proved only 20 years later by Bozejko and Speicher. Since then many people studied operator algebras generated by q-Gaussian random variables.

q-Gaussian algebras: the construction

Let us have a look at the construction!

The case $q = 0$

First consider the case where $q = 0$. For this, fix a Hilbert space $\mathcal{H}$.

- **Step 1.** Form the *full Fock space* $\mathcal{F}_0(\mathcal{H}) := \bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}$ where $\mathcal{H}^{\otimes 0} := \mathbb{C}\Omega$ with the *vacuum vector* Ω.

- **Step 2.** For $\xi \in \mathcal{H}$ define operators $l_0^\dagger(\xi)$ and $l_0(\xi)$ on $\mathcal{F}_0(\mathcal{H})$ by

$$
\begin{aligned}
l_0^\dagger(\xi) \quad &: \quad \Omega \mapsto \xi, \\
&\quad\ \eta_1 \otimes \ldots \otimes \eta_n \mapsto \xi \otimes \eta_1 \otimes \ldots \otimes \eta_n \\
l_0(\xi) \quad &: \quad \Omega \mapsto 0, \\
&\quad\ \eta_1 \otimes \ldots \otimes \eta_n \mapsto \langle \xi, \eta_1 \rangle\, \eta_2 \otimes \ldots \otimes \eta_n.
\end{aligned}
$$

One checks that $l_0^\dagger(\xi), l_0(\xi) \in \mathcal{B}(\mathcal{F}_0(\mathcal{H}))$, $l_0^\dagger(\xi) = (l_0(\xi))^*$ and that indeed $l_0(\xi) l_0(\eta)^* = \langle \xi, \eta \rangle \,\mathrm{id}$ for all $\xi, \eta \in \mathcal{H}$.

The general case

Idea: To realize general q-relations, perform the same construction, but "deform" the inner product of the Fock space.

Fix $q \in [-1, 1]$ and let $\mathcal{H}$ be a Hilbert space. Let us (heuristically!) deduce how the suitable construction should look like!

- **Step 1.** Consider the (purely algebraic) vector space $F(\mathcal{H}) := \mathrm{alg} - \bigoplus_{n \geq 0} \mathcal{H}^{\odot n}$.

- **Step 2.** For $\xi \in \mathcal{H}$ define $l_q^\dagger(\xi)$ as before (i.e. $l_q^\dagger(\xi)(\eta_1 \otimes \ldots \otimes \eta_n) := \xi \otimes \eta_1 \otimes \ldots \otimes \eta_n$ and $l_q^\dagger(\xi)\Omega := \xi$) and set $l_q(\xi)\Omega = 0$.

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As before, we want the operators to satisfy $l_q^\dagger(\xi) = (l_q(\xi))^*$ and the q-relations (with respect to a suitable "deformed" inner product $\langle\cdot,\cdot\rangle_q$). Thus it has to follow that

$$\begin{aligned} l_q(\xi)(\eta_1 \otimes \ldots \otimes \eta_n) &= l_q(\xi) l_q^\dagger(\eta_1)(\eta_2 \otimes \ldots \otimes \eta_n) \\ &= \langle \xi, \eta_1 \rangle_q \eta_2 \otimes \ldots \otimes \eta_n + q l_q^\dagger(\eta_1) l_q(\xi)(\eta_2 \otimes \ldots \otimes \eta_n) \\ &= \ldots \\ &= \sum_{k=1}^{n} q^{k-1} \langle \xi, \eta_k \rangle_q \eta_1 \otimes \ldots \otimes \widehat{\eta}_k \otimes \ldots \otimes \eta_n. \qquad (1) \end{aligned}$$

- **Step 3.** The formula (1) motivates to define the inner product $\langle \cdot, \cdot \rangle_q$ inductively via $\langle \Omega, \Omega \rangle_q := 1$, $\langle \eta_1 \otimes \ldots \otimes \eta_m, \xi_1 \otimes \ldots \otimes \xi_n \rangle_q = 0$ if $m \neq n$ and

$$
\begin{aligned}
& \langle \eta_1 \otimes \ldots \otimes \eta_n, \xi_1 \otimes \ldots \otimes \xi_n \rangle_q \\
= \quad & \left\langle \eta_1 \otimes \ldots \otimes \eta_n, l_q^\dagger(\xi_1) \xi_2 \otimes \ldots \otimes \xi_n \right\rangle_q \\
= \quad & \langle l_q(\xi_1) \eta_1 \otimes \ldots \otimes \eta_n, \xi_2 \otimes \ldots \otimes \xi_n \rangle_q \\
= \quad & \sum_{k=1}^{n} q^{k-1} \langle \xi_1, \eta_k \rangle_q \langle \eta_1 \otimes \ldots \otimes \widehat{\eta}_k \otimes \ldots \otimes \eta_n, \xi_2 \otimes \ldots \otimes \xi_n \rangle_q .
\end{aligned}
$$

One checks that for $q \in (-1, 1)$ this defines a positive definite inner product on $F(\mathcal{H})$.

- **Step 4.** Complete $F(\mathcal{H})$ with respect to $\langle \cdot, \cdot \rangle_q$ and call this the *q-deformed Fock space* $\mathcal{F}_q(\mathcal{H})$. Then the operators $l_q(\xi) \in \mathcal{B}(\mathcal{F}_q(\mathcal{H}))$, $\xi \in F(\mathcal{H})$ do what we want.

Remark: The definition of $\langle \cdot, \cdot \rangle_q$ can be made more explicit.

Operator algebras generated by $l_q(\xi)$

We are now interested in operator algebras induced by creation and annihilation operators.

Proposition

The von Neumann algebra in $\mathcal{B}(\mathcal{F}_q(\mathcal{H}))$ generated by $\{l_q(\xi) \mid \xi \in \mathcal{H}\}$ is $\mathcal{B}(\mathcal{F}_q(\mathcal{H}))$.

This gives pretty boring operator algebras. So we adapt the setting.

Definition

Let $\mathcal{H}_\mathbb{R}$ be a real (separable) Hilbert space and denote by $\mathcal{H} := \mathcal{H}_\mathbb{R} \oplus i\mathcal{H}_\mathbb{R}$ its complexification. Fix $q \in (-1, 1)$.

- The $*$-algebra

$$\mathcal{A}_q(\mathcal{H}_\mathbb{R}) := * - \mathrm{alg}\{l_q(\xi) + l_q(\xi)^* \mid \xi \in \mathcal{H}_\mathbb{R}\} \subseteq \mathcal{B}(\mathcal{F}_q(\mathcal{H}))$$

  is called the *q-Gaussian algebra*;
- The $*$-algebra

$$A_q(\mathcal{H}_\mathbb{R}) := \overline{\mathcal{A}_q(\mathcal{H}_\mathbb{R})}^{\|\cdot\|} \subseteq \mathcal{B}(\mathcal{F}_q(\mathcal{H}))$$

  is called the *q-Gaussian C^*-algebra*;
- The $*$-algebra

$$M_q(\mathcal{H}_\mathbb{R}) := \mathcal{A}_q(\mathcal{H}_\mathbb{R})'' \subseteq \mathcal{B}(\mathcal{F}_q(\mathcal{H}))$$

  is called the *q-Gaussian von Neumann algebra.*

The algebras $\mathcal{A}_q(\mathcal{H}_{\mathbb{R}})$, $A_q(\mathcal{H}_{\mathbb{R}})$ and $M_q(\mathcal{H}_{\mathbb{R}})$ are all tracial: the vector state $\tau_\Omega : x \mapsto \langle x\Omega, \Omega \rangle_q$ defines a faithful normal tracial state.

For $q = 0$ the q-Gaussian von Neumann algebras further coincide with the *free group factors*, i.e.

$$M_0(\mathcal{H}_{\mathbb{R}}) \cong \mathcal{L}(\mathbb{F}_{\dim(\mathcal{H}_{\mathbb{R}})}).$$

What is known about these operator algebras?

The q-Gaussians have been studied for several years, mainly on the von Neumann algebraic level.

- *Ricard, 2005*: If $\dim(\mathcal{H}_\mathbb{R}) \geq 2$, then $M_q(\mathcal{H}_\mathbb{R})$ is a *factor* for all $q \in (-1, 1)$, i.e. $\{x \in M_q(\mathcal{H}_\mathbb{R}) \mid xy = yx \text{ for all } y \in M_q(\mathcal{H}_\mathbb{R})\} = \mathbb{C}1$;

- *Nou, 2004*: If $\dim(\mathcal{H}_\mathbb{R}) \geq 2$, then $M_q(\mathcal{H}_\mathbb{R})$ is *non-injective* for all $q \in (-1, 1)$, i.e. there exists no conditional expectation $\mathcal{B}(\mathcal{F}_q(\mathcal{H}_\mathbb{R})) \to M_q(\mathcal{H}_\mathbb{R})$;

- *Avsec, 2011*: If $2 \leq \dim(\mathcal{H}_\mathbb{R}) < \infty$, then $M_q(\mathcal{H}_\mathbb{R})$ is *strongly solid* for all $q \in (-1, 1)$. In particular, $M_q(\mathcal{H}_\mathbb{R})$ is *prime* (i.e. it can not decomposed into a tensor product of “large” von Neumann algebras) and does not contain *Cartan subalgebras* (i.e. it does not contain “large” commutative subalgebras).

Approximation properties:

- *Folklore*: The q-Gaussians $M_q(\mathcal{H}_{\mathbb{R}})$ satisfy the *Haagerup approximation property*;
- *Avsec, 2011* and *Wasilewski, 2020*: The q-Gaussians $M_q(\mathcal{H}_{\mathbb{R}})$ satisfy the *weak-$*$ complete metric approximation property*.

What about the dependence on the parameter q? On the algebraic level:

Theorem (Caspers-Isono-Wasilewski, 2021)

For all $q, q' \in (-1, 1)$, $\mathcal{A}_q(\mathcal{H}_\mathbb{R}) \cong \mathcal{A}_{q'}(\mathcal{H}_\mathbb{R})$.

The question becomes much harder on the C*-algebraic and von Neumann algebraic level:

Theorem (Guionnet-Shlyakhtenko, 2014)

Assume that $\dim(\mathcal{H}_\mathbb{R}) < \infty$. Then there exists $q_0 > 0$ such that

$$A_q(\mathcal{H}_\mathbb{R}) \cong A_0(\mathcal{H}_\mathbb{R}) \qquad \text{and} \qquad M_q(\mathcal{H}_\mathbb{R}) \cong M_0(\mathcal{H}_\mathbb{R})$$

for all $q \in (-1, 1)$ with $|q| < q_0$.

A similar result holds for certain infinitely generated "mixed" q-Gaussians (Nelson-Zeng, 2018).

One recent result that should be mentioned:

Theorem (Kuzmin, 2022)

For $\dim(\mathcal{H}_{\mathbb{R}}) < \infty$ the isomorphism class of

$$C^*\left(\{l_q(\xi) \mid \xi \in \mathcal{H}_{\mathbb{R}}\}\right) \subseteq \mathcal{B}(\mathcal{F}_q(\mathcal{H}))$$

does not depend on $q \in (-1, 1)$.

This extends earlier results by Jorgensen-Schmitt-Werner, 1995 and Dykema-Nica, 1993.

Two natural questions

Question (Nelson-Zeng, 2018)

Assume that $\dim(\mathcal{H}_\mathbb{R}) < \infty$. Is it true that

$$A_q(\mathcal{H}_\mathbb{R}) \cong A_0(\mathcal{H}_\mathbb{R}) \qquad \text{and} \qquad M_q(\mathcal{H}_\mathbb{R}) \cong M_0(\mathcal{H}_\mathbb{R})$$

for all $q \in (-1,1)$?

Question

What about the case where $\dim(\mathcal{H}_\mathbb{R}) = \infty$?

How to approach these questions (in the C^*-algebraic case)?

The Akemann-Ostrand property

Let A be a C*-algebra. The opposite C*-algebra, denoted by A^{op}, is A (as a normed, involutive, linear space) but equipped with the multiplication $a \cdot b := ba$.

Definition

A von Neumann algebra M equipped with a tracial state τ has the *Akemann-Ostrand property* (or *property AO*), if there exists a σ-weakly dense unital C*-algebra $A \subseteq M$ such that A is locally reflexive and such that the canonical map

$$\theta : A \odot A^{\text{op}} \to \mathcal{B}(L^2(M))/\mathcal{K}(L^2(M)), a \otimes b^{\text{op}} \mapsto ab^{\text{op}} + \mathcal{K}(L^2(M))$$

is continuous with respect to the minimal tensor norm. Here A^{op} acts on $L^2(M)$ via "right multiplication".

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Example

(1) A von Neumann algebra M equipped with a faithful normal tracial state τ is injective if and only if

$$\theta : M \odot M^{\mathrm{op}} \to \mathcal{B}(L^2(M)), a \otimes b^{\mathrm{op}} \mapsto ab^{\mathrm{op}}$$

is continuous with respect to the minimal tensor norm.

(2) Let $n \geq 2$. Then $\mathcal{L}(\mathbb{F}_n)$ has the Akemann-Ostrand property with respect to $C_r^*(\mathbb{F}_n) \subseteq \mathcal{L}(\mathbb{F}_n)$ (but $\mathcal{L}(\mathbb{F}_n)$ is not injective).

The isomorphism problem in the C*-algebraic case

Do the q-Gaussians $M_q(\mathcal{H}_\mathbb{R})$ have the Akemann-Ostrand property (with respect to $A_q(\mathcal{H}_\mathbb{R})$)?

Theorem (Shlyakhtenko, 2004 and Caspers-Isono-Wasilewski, 2021)

Assume that $\dim(\mathcal{H}_\mathbb{R}) < \infty$ and let $q \in (-1,1)$. Then $M_q(\mathcal{H}_\mathbb{R})$ (equipped with τ_Ω) has the Akemann-Ostrand property with respect to $A_q(\mathcal{H}_\mathbb{R}) \subseteq M_q(\mathcal{H}_\mathbb{R})$ if $|q| < \sqrt{2} - 1$ or $|q| \leq (\dim(\mathcal{H}_\mathbb{R}))^{-1/2}$.

What about the case $\dim(\mathcal{H}_\mathbb{R}) = \infty$?

Proposition (Houdayer, 2007)

Assume that $\dim(\mathcal{H}_\mathbb{R}) = \infty$. Then $M_0(\mathcal{H}_\mathbb{R})$ (equipped with τ_Ω) has the Akemann-Ostrand property with respect to $A_0(\mathcal{H}_\mathbb{R}) \subseteq M_0(\mathcal{H}_\mathbb{R})$.

Proposition (Borst-Caspers-K.-Wasilewski, 2022)

Assume that $\dim(\mathcal{H}_{\mathbb{R}}) = \infty$ and let $q \in (-1,1) \setminus \{0\}$. Then $M_q(\mathcal{H}_{\mathbb{R}})$ (equipped with τ_Ω) does not have the Akemann-Ostrand property with respect to $A_q(\mathcal{H}_{\mathbb{R}}) \subseteq M_q(\mathcal{H}_{\mathbb{R}})$.

Warning: This does not (yet) imply that $A_q(\mathcal{H}_{\mathbb{R}}) \ncong A_0(\mathcal{H}_{\mathbb{R}})$ for $\dim(\mathcal{H}_{\mathbb{R}}) = \infty$ and $q \in (-1,1) \setminus \{0\}$.

The missing ingredient is the trace-uniqueness:

Theorem (Borst-Caspers-K.-Wasilewski, 2022)

Assume that $\dim(\mathcal{H}_{\mathbb{R}}) = \infty$ and let $q \in (-1,1)$. Then $A_q(\mathcal{H}_{\mathbb{R}})$ carries τ_Ω as its unique tracial state. Further, $A_q(\mathcal{H}_{\mathbb{R}})$ is simple (i.e. it contains no non-trivial two-sided ideals).

The theorem strengthens the factoriality result from before.

The proof uses a very classical averaging argument (in combination with certain inequalities by Nou). The assumption $\dim(\mathcal{H}_{\mathbb{R}}) = \infty$ is crucial.

Corollary (Borst-Caspers-K.-Wasilewski, 2022)

Assume that $\dim(\mathcal{H}_{\mathbb{R}}) = \infty$ and let $q \in (-1,1) \setminus \{0\}$. Then $A_q(\mathcal{H}_{\mathbb{R}}) \ncong A_0(\mathcal{H}_{\mathbb{R}})$.

Sketch of the proof: Assume that $A_q(\mathcal{H}_{\mathbb{R}}) \cong A_0(\mathcal{H}_{\mathbb{R}})$. Then $M_q(\mathcal{H}_{\mathbb{R}}) \cong M_0(\mathcal{H}_{\mathbb{R}})$ by the trace-uniqueness. But then, since $M_0(\mathcal{H}_{\mathbb{R}})$ has property (AO) with respect to $A_0(\mathcal{H}_{\mathbb{R}})$, $M_q(\mathcal{H}_{\mathbb{R}})$ has property (AO) with respect to $A_q(\mathcal{H}_{\mathbb{R}})$. This contradicts our earlier results. ■

Similarly: $A_q(\mathcal{H}_{\mathbb{R}})$ is not isomorphic to $A_{q'}(\mathbb{R}^d)$ with $d \in \mathbb{N}$, $q' \in (-1,1)$.

Thank you!