α-induction and bi-unitary connections

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Subfactors, fusion categories, braiding and connections

α-induction is a functorial method to produce a new fusion category from a C^*-braided fusion category and a Frobenius algebra in it. It has been studied for fusion categories of endomorphisms and bimodules arising from subfactors, particularly in conformal field theory. A family of (bi-unitary) connections also describes a fusion category. We describe α-induction in terms of connections.

Outline of the talk:

1. A braided fusion category and α-induction
2. Commuting squares and connections
3. α-induction for connections

Fusion categories

A finite group G consists of the following ingredients.

1. An associative multiplication
2. The identity element
3. The inverse elements

For a finite group G, the set of its finite dimensional unitary representations have the following structures.

1. Irreducible decomposition into finitely many ones
2. An associative tensor product
3. The identity representation
4. The dual representation

An abstract axiomatization of this structure gives a fusion category.

Endormophisms, bimodules and fusion categories

Let M be a factor of type III. A (unital $*$-)endomorphism λ with $[M : \lambda(M)] < \infty$ gives an object of a tensor category. The tensor product operation is given by composition of endomorphisms. The conjugate endomorphism $\bar{\lambda}$ is given in terms of the modular conjugation operators arising from the Tomita-Takesaki theory. A closed system of finitely many irreducible endomorphisms gives a fusion category.

Let M be a factor of type II_1. An M-M bimodule H with $\dim_M H < \infty$, $\dim H_M < \infty$ gives an object of a tensor category. The tensor product operation is given by a relative tensor product over M of bimodules. The dual bimodule $\bar{H}$ is naturally defined.

Braiding on fusion categories

For two representations π and σ of a group G, the two tensor products $\pi \otimes \sigma$ and $\sigma \otimes \pi$ are trivially unitarily equivalent. The corresponding equivalence for a general fusion category is not assumed.

We have an important class of fusion categories for which the above commutativity of tensor products holds in some mathematically nice way. Such commutativity is called braiding because it is similar to switching two wires.

A braiding naturally comes in a pair — overcrossing and undercrossing. It is more interesting if these two are really different. If this is the case, the fusion category is called a modular tensor category.

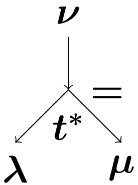
Conformal field theory and braiding

A 2-dimensional conformal field theory is a quantum field theory with conformal symmetry. It splits into two chiral halves and each lives on S^1, a compactified light ray. In algebraic quantum field theory, we consider a conformal net $\{A(I)\}_{I\subset S^1}$ where I is an interval in the circle. Each $A(I)$ is a von Neumann algebra generated by observables in I, and automatically a hyperfinite type III_1 factor under the standard set of axioms.

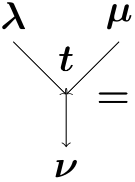
Finite dimensional unitary representations of a conformal net give a braided category of Doplicher-Haag-Roberts superselection sectors. If we have only finitely many irreducible representations, we get a modular tensor category (K-Longo-Müger).

Braided fusion category and graphical calculus

Suppose we have a braided fusion category. An (irreducible) object in the category is represented with a wire. When two wires $\boldsymbol{\lambda}, \boldsymbol{\mu}$ merge into a wire $\boldsymbol{\nu}$, the triple point represents an isometric intertwiner $t \in \mathrm{Hom}(\lambda\mu, \nu)$ with an appropriate normalization, where d stands for the dimension of an irreducible object.



$= \sqrt[4]{\frac{d_\lambda d_\mu}{d_\nu}}\, t^*,$



$= \sqrt[4]{\frac{d_\lambda d_\mu}{d_\nu}}\, t.$

Note that we read our diagram from the top to the bottom, while some authors use the opposite convention.

A Frobenius algebra in a braided fusion category

Suppose we have a subfactor $N \subset M$ with finite index and finite depth. The latter condition means irreducible decompositions of tensor powers of the bimodule ${}_N M_N$ give only finitely many irreducible N-N bimodule up to isomorphism. These N-N bimodules give a fusion category. The bimodule ${}_N M_N$ gives a so-called algebra in a fusion category.

Such an algebra in our setting is called a Frobenius algebra. If the fusion category is braided, we have a notion of a commutative Frobenius algebra. For a modular tensor category arising from a chiral conformal field theory, a commutative Frobenius algebra corresponds to its extension.

α-induction for endomorphisms

If we have a (not necessarily commutative) Frobenius algebra in a braided fusion category of endomorphisms of N corresponding to a subfactor $N \subset M$, then each endomorphism λ in the fusion category has an extension $\alpha_\lambda^\pm$ to M depending on the choice of a braiding as follows.

$$\alpha_\lambda^\pm = \bar{\iota}^{-1} \cdot \mathrm{Ad}(\varepsilon^\pm(\lambda, \theta)) \cdot \lambda \cdot \bar{\iota}$$

Here ι is the inclusion map of N into M and we have $M = Nv$ with a nice isometry v. We have $\alpha_\lambda^\pm(x) = \lambda(x)$ for $x \in N$ and $\alpha_\lambda^\pm(v) = \varepsilon^\pm(\lambda, \theta)^* v$. This was first defined by Longo-Rehren and studied by Xu, Böckenhauer-Evans, and Böckenhauer-Evans-K.

α-induction for bimodules

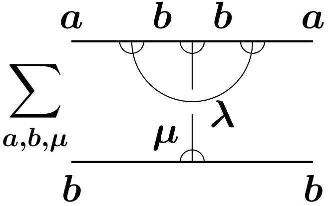
For a braided fusion category of bimodules over N, we have an even simpler description of α-induction.

Suppose we have an N-N bimodule ${}_N X_N$ in the fusion category arising from a subfactor $N \subset M$. Braiding gives an isomorphism ${}_N M \otimes_N X_N \cong {}_N X \otimes_N M_N$. The left hand side has a natural left action of M and the right hand side has a natural right action of M. We can show that these two actions commute after identification of the both hand sides and we have an M-M bimodule. This is the α-induced bimodule. Note that this bimodule depends on the choice of a braiding.

We can also show that this assignment of an M-M bimodule is functorial.

α-induction and Ocneanu's graphical calculus

Ocneanu introduced a graphical expression for an M-M bimodule in his double triangle algebra, which gives a general description of M-M bimodules using N-N and N-M bimodules, as follows.



Here a thin wire represents an N-N bimodule and a thick wire represents an N-M or M-N bimodule. This construction is essentially the same as α-induction.

α-induction and modular invariants

We set $Z_{\lambda,\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$, where λ, μ are irreducible objects of the braided fusion category.
Böckenhanuer-Evans-K showed that Z commutes with the S- and T-matrices arising from the braiding structure of the braided fusion category. Each $Z_{\lambda,\mu}$ is a non-negative integer and we have $Z_{0,0} = 1$ where 0 denotes the identity object in the fusion category.

Such a matrix Z for a modular tensor category is called a modular invariant and has been studied well in mathematical physics and classified for several concrete modular tensor categories such as $SU(N)_k$ for small N.

The Frobenius algebra object is given by $\bigoplus_\lambda Z_{\lambda,0}\lambda$, if it is commutative.

α-induction in conformal field theory

Let $\{A(I)\}_{I\subset S^1}$ be a conformal net. Suppose it has only finitely many irreducible representations. Its representation category is a modular tensor category. Commutative Frobenius algebras in it are in a bijective correspondence to conformal nets extending $\{A(I)\}_{I\subset S^1}$.

Fix an extension $\{B(I)\}_{I\subset S^1}$ of $\{A(I)\}_{I\subset S^1}$. The irreducible objects that simultaneously arise from both positive and negative α-inductions exactly correspond to irreducible representations of $\{B(I)\}_{I\subset S^1}$.

The Frobenius algebra for $\{B(I)\}_{I\subset S^1}$ is recovered from the modular invariant Z as in the previous slide.

A commuting square

Consider $\begin{array}{ccc} A & \subset & B \\ \cap & & \cap \\ C & \subset & D \end{array}$ where A, B, C, D are finite dimensional C^*-algebras with a trace on D. We say this is a commuting square if the restriction to C of the conditional expectation E_B from D to B is equal to the conditional expectation E_A from C to A.

In order to avoid some not-so-interesting examples, we require that BC, the span of the products bc with $b \in B$ and $c \in C$, is equal to D. Such a commuting square is said to be nondegenerate. In this talk, a commuting square means a finite dimensional nondegenerate commuting square.

A (bi-unitary) connection

For a choice of one edge each from the four Bratteli diagrams of a commuting square, the connection W gives a complex number to each such square with the following, which is bi-unitarity.

$$\sum_{z,\xi_1,\xi_2} \begin{array}{ccc} x & \xi_4 & y \\ \xi_1 & \boxed{W} & \xi_3 \\ z & \xi_2 & w \end{array} \; \overline{\begin{array}{ccc} x & \xi'_4 & y' \\ \xi_1 & \boxed{W} & \xi'_3 \\ z & \xi_2 & w \end{array}} = \delta_{\xi_3,\xi'_3}\delta_{\xi_4,\xi'_4}$$

$$\begin{array}{ccc} y & \tilde{\xi}_4 & x \\ \xi_3 & \boxed{W'} & \xi_1 \\ w & \tilde{\xi}_2 & z \end{array} = \sqrt{\frac{\mu_x\mu_w}{\mu_y\mu_z}} \; \overline{\begin{array}{ccc} x & \xi_4 & y \\ \xi_1 & \boxed{W} & \xi_3 \\ z & \xi_2 & w \end{array}}$$

An example of a connection on the Dynkin diagrams

We give an example of a connection as follows. Fix one of the A-D-E Dynkin diagram and use it for the four Bratteli diagrams. Let n be its Coxeter number and set $\varepsilon = \sqrt{-1}\exp\frac{\pi\sqrt{-1}}{2(n+1)}$. We write μ_x for the Perron-Frobenius eigenvector entry for a vertex x. Then our connection is given as follows.

$$\begin{matrix} j & & k \\ & \boxed{W} & \\ l & & m \end{matrix} = \delta_{kl}\varepsilon + \sqrt{\frac{\mu_k\mu_l}{\mu_j\mu_m}}\delta_{jm}\bar{\varepsilon}$$

This is similar to a Boltzmann weight for a lattice model.

From a connection to a subfactor

We start with a commuting square and repeat the basic constructions horizontally.

$$\begin{array}{ccccccccc} A_{00} & \subset & A_{01} & \subset & A_{02} & \subset & A_{03} & \subset & \cdots \\ \cap & & \cap & & \cap & & \cap & & \\ A_{10} & \subset & A_{11} & \subset & A_{12} & \subset & A_{13} & \subset & \cdots \end{array}$$

This gives a sequence of commuting squares. The GNS-completions of $\bigcup_{n=1}^{\infty} A_{0n} \subset \bigcup_{n=1}^{\infty} A_{1n}$ with respect to trace give a hyperfinite type II_1 subfactor $A_{0,\infty} \subset A_{1,\infty}$ of finite index. The vertical basic constructions give finite dimensional C^*-algebras A_{kn} with trace and we have the Jones tower:

$$A_{0,\infty} \subset A_{1,\infty} \subset A_{2,\infty} \subset A_{3,\infty} \subset \cdots .$$

From a subfactor to a connection

We start with a subfactor $N \subset M$ of finite index. We then have the Jones tower

$$N \subset M \subset M_1 \subset M_2 \subset M_3 \subset \cdots .$$

The higher relative commutants $N' \cap M_k$ are finite dimensional C^*-algebras. If the subfactor $N \subset M$ is of finite depth, then the following gives a canonical commuting square for sufficiently large k.

$$\begin{array}{ccc} M' \cap M_k & \subset & M' \cap M_{k+1} \\ \cap & & \cap \\ N' \cap M_k & \subset & N' \cap M_{k+1} \end{array}$$

By Popa's classification theorem, this fully recovers the original subfactor $N \subset M$, if it is hyperfinite.

A flat connection

The connection arising from a subfactor as in the previous slide has a particulary nice property compared with general connections. This property is called flatness. Flatness is characterized as certain mutual commutativity of two finite dimensional C^*-algebras. It can be also formulated in terms of the value of a large diagram given by the connection. It is easy to compute the value of such a diagram numerically with a computer, but a rigorous proof is much more difficult.

Among the connections on the A-D-E Dynkin diagrams, those on A_n, D_{2n}, E_6 and E_8 are flat, and those on D_{2n+1} and E_7 are not. So only the former ones appear in classification of subfactors.

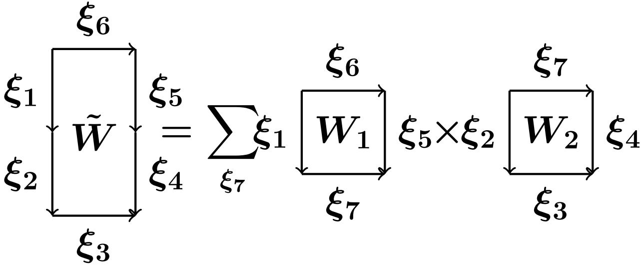
An open string bimodule

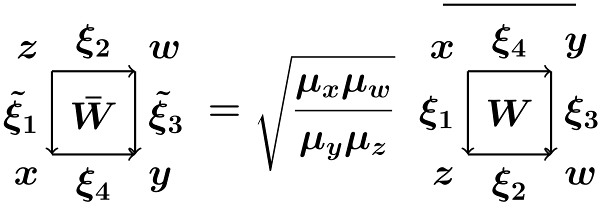
We can construct a subfactor $N \subset M$ from a connection and it gives a bimodule ${}_N M_M$. We can also construct a bimodule directly from a connection. This is called an open string bimodule due to Ocneanu and Asaeda-Haagerup. For this construction, we can translate a relative tensor product of open string bimodules, a dual open string bimodule, an isomorphism of open string bimodules and intertwiners between open string bimodules into the language of connections.

These are infinite dimensional problems in the setting of bimodules, but they are finite dimensional problems in the setting of connections, and we can compute these in terms of finite dimensional matrices on computers.

A tensor product of connections and a dual connection

We have the tensor product and the dual of connections.





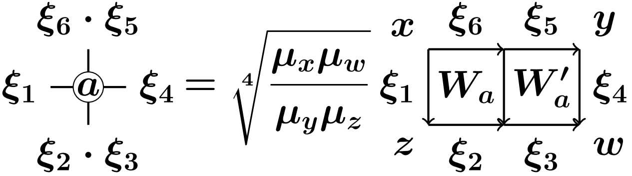
Intertwiners between connections

By Ocneanu's compactness argument, the problems of an isomorphism of open string bimodules and intertwiners between open string bimodules are reduced to finite dimensional problems in linear algebra. This is an advantage of the connection approach.

In particular, two open string bimodules are isomorphic if and only if the graphs of the two connections are the same and the two connections are equivalent up to vertical gauge choices. An intertwiner is also described with a flat field of string and it gives an analogue of vertical gauges in turn. Everything is described with linear algebra, though practical computations are rather hard even on computers.

Relations to 2-dimensional statistical physics

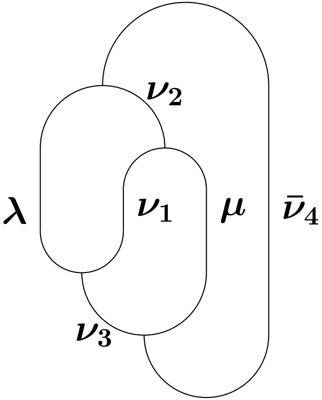
A ***k*-tensor** is a (finite) family of complex numbers indexed by k indices. A connection is determined by four edges, so it gives a 4-tensor as in the following figure. Here W'_a is a horizontal dual of W_a.



Such a 4-tensor has been used recently by physicists in studies of 2-dimensional topological order and they are interested in explicit computations using computers.

A fusion category of endomorphisms and a connection

Suppose we have a fusion category of endomorphisms and $\lambda, \mu, \nu_1, \nu_2, \nu_3, \nu_4$ are irreducible objects. For fixed λ, μ, the following diagram gives a (flat) connection.



($6j$-symbols)

An α-induced connection

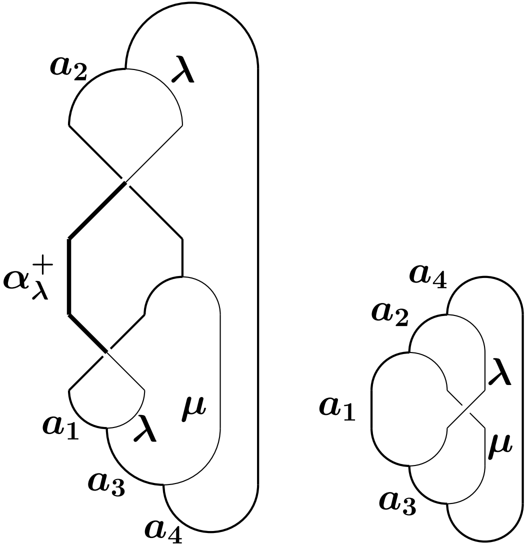
We introduce the α-induced connection as in the following figure.

$$\begin{array}{ccc} & \alpha^+_\lambda a_1\mu \xrightarrow{\alpha^+_\lambda(T_1)} \alpha^+_\lambda a_2 & \\ \mathcal{E}^+(\lambda, a_1)\downarrow & & \downarrow \mathcal{E}^+(\lambda, a_2) \\ a_1\lambda\mu & & a_2\lambda \\ T_3'\downarrow & & \downarrow T_2' \\ & a_3\mu \xrightarrow[T_4]{} a_4 & \end{array}$$

This is a rewriting of a simple diagram using braiding operators.

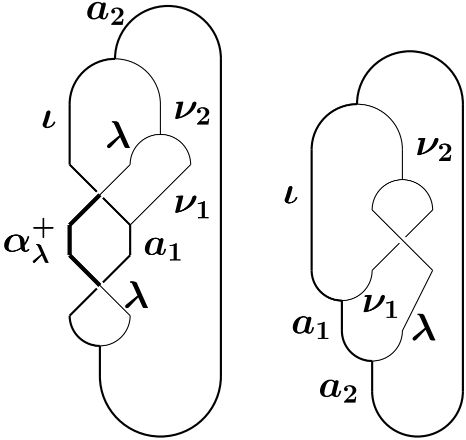
Redrawing of the α-induced connection

We next redraw the figure in the previous slide as follows.



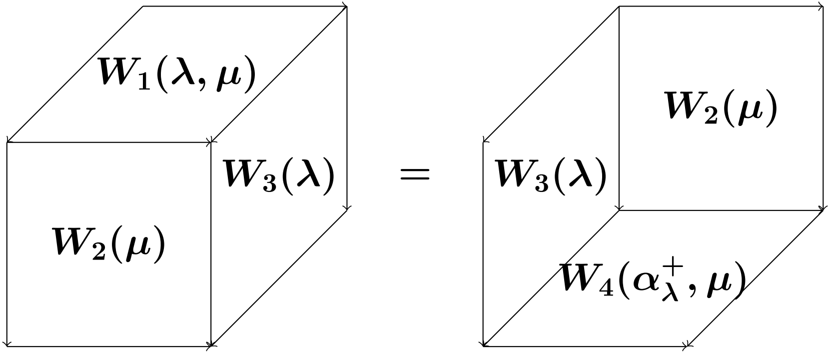
Intertwining connections between connections

We also have an intertwining connection between the original connection and the α-induced connection as in the following figures.



The intertwining Yang-Baxter equation

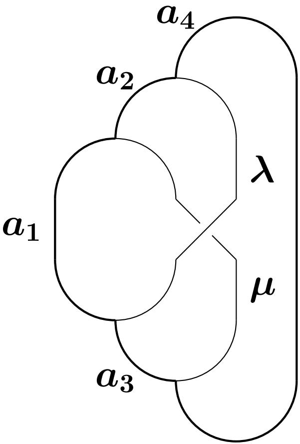
Among these connections, we have the following identity, the intertwining Yang-Baxter equation.

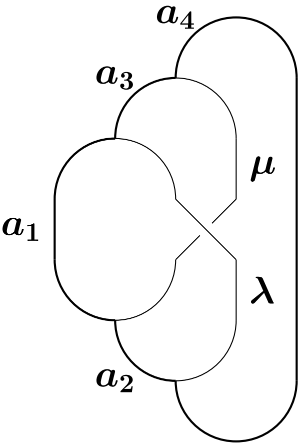


Both hand sides represent summations of product of three connection values.

Switching braiding operators and complex conjugate

We now study the relation between α^+-induced connections and α^--induced connections.





Examples of A-D-E Dynkin diagrams

Suppose the original modular tensor category is the one arising from the Wess-Zumino-Witten model $SU(2)_k$. In this case, the Frobenius algebras are well-understood. We have commutative ones corresponding to A_n, D_{2n}, E_6 and E_8, and noncommutative ones corresponding to the D_{2n+1} and E_7. The ones for D_{2n} correspond to simple current extensions and the ones for E_6 and E_8 correspond to conformal embeddings. α-induction for them is also well-understood.

Connections arising from this α-induction are exactly those on the A-D-E Dynkin diagrams. They also produce more connections generated by them. This is the setting Ocneanu originally worked on.

A triple sequence of string algebras

In this setting, we can construct a triple sequence of string algebras, certain finite dimensional C^*-algebras arising from the bipartite graphs. Note that the standard setting of string algebras and a connection gives a double sequence of string algebras. The intertwining Yang-Baxter equation is compatibility of two identifications of operators for such a triple sequence. This setting was studied by myself under the name of paragroup actions on subfactors.

Such a triple sequence gives a commuting square of hyperfinite type II_1 factors. They correspond to commuting squares of type III factors arising from α-induction for endomorphisms.