A phase transition for tails of the free multiplicative convolution powers

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Introduction

General idea

S-transform

- For probability measure μ on $[0, +\infty)$ denote $\delta = \mu(0) < 1$. The moment transform is defined as
$$\psi_\mu(z) = \int_{[0,+\infty)} \frac{zt}{1 - zt} d\mu(t).$$
- $\psi_\mu : (-\infty, 0) \to (\delta - 1, 0)$ is invertible, denote the inverse χ_μ.
- The S-transform is defined as
$$S_\mu(z) = \frac{z+1}{z} \chi_\mu(z).$$
- For this talk we will view the S-transform as a real function.

General idea

Free multiplicative convolution

- For μ, ν probability measures on $[0, +\infty)$ the free multiplicative convolution is well defined and denoted $\mu \boxtimes \nu$.
- We have
$$S_{\mu\boxtimes\nu}(z) = S_{\mu}(z)S_{\nu}(z)$$
for $z \in (-\varepsilon, 0)$, for some $\varepsilon > 0$.
- The above determines uniquely the free multiplicative convolution.
- On the other hand due to the compositional inverse involved when calculating ψ it is hard to get explicit examples.

General idea

Problem
Can we get some information about the measure directly from the S-transform, without referring to ψ?

Our framework

- We are interested in the tail behavior, i.e. for a probability measure μ on $[0, +\infty)$ we look at the behavior at $+\infty$ of the function

$$\overline{\mu}(x) = \mu((x, +\infty)).$$

- We focus on measures with regularly varying tail (explained later).
- For such measures we are able to determine explicitly the tail behavior **directly** from the S-transform.

Related results

- Hazra, Maulik, *Free subexponentiality*, AoP (2013)
  - Description of behavior of the R-transform for probability measures with regularly varying tails.
  - Authors extended and used the key ingredient from Bercovici, Pata & Biane, Ann. Math. (1999): If $|z| \to \infty$ with $\Re z / \Im z$ is bounded we have

$$\phi_\mu(z) := F_\mu^{-1}(z) - z \sim z^2 \left(G_\mu(z) - \frac{1}{z} \right),$$

    where $G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} \mu(dt)$ and $F_\mu = 1/G_\mu$,
- Chakraborty, Hazra, *Boolean convolutions and regular variation*, ALEA (2018)
  - Description of behavior of transforms relevant to Boolean additive and multiplicative convolutions.
  - Boolean case is rather easy, because there are no inverses of the Cauchy transform involved.

Slowly varying functions
If $L\colon (0,\infty) \to (0,\infty)$ satisfies

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1 \qquad \text{for all } \lambda > 0,$$

then we say that L is slowly varying. We denote it by $L \in \mathcal{R}_0$.

Example (of $L \in \mathcal{R}_0$)

- $L(x) = c$ *for some* $c \in \mathbb{R}^+$.
- $L(x) = \prod_{k=1}^{n} (\log^{(k)}(x))^{\alpha_k}$, *where* $\log^{(k)}$ *is the* k*th iterate of* $\log$.

Regularly varying tails

Definition

- *If* $f(x) = x^{\rho} L(x)$ *for some* $L \in \mathcal{R}_0$ *and* $\rho \in \mathbb{R}$ *we say that* f *is regularly varying with index* ρ *and write* $f \in \mathcal{R}_{\rho}$.
- *A measure* μ *is said to have regularly varying right tail if*

$$x \mapsto \bar{\mu}(x) := \mu\big((x, \infty)\big) \in \mathcal{R}_{-\alpha}$$

for some $\alpha \geq 0$.

Tails vs S–transform

Overview

- One of our goals is to develop tools for the study of the tails of free multiplicative convolution.
- The usual tool to study $\boxtimes$ is Voiculescu's S-transform, for which we have

$$S_{\mu \boxtimes \nu} = S_\mu S_\nu.$$

- We will characterize behavior of the S-transform of a probability measure with regularly varying tails.

Notation

Let $\mathcal{M}_+$ denote the set of Borel probability measures on $\mathbb{R}_+ = [0, \infty)$.

$$\mathcal{M}_p = \{\mu \in \mathcal{M}_+ \colon m_p(\mu) < \infty \text{ and } m_{p+1}(\mu) = \infty\}.$$

Important formula: denote the p-th moment of μ by $m_p(\mu)$ then

$$m_p(\mu) = p \int_0^\infty t^{p-1} \overline{\mu}(t) dt.$$

Notation
$f \sim g$ means $f(x)/g(x) \to 1$ as $x \to \infty$

Idea behind 1

- If $\mu \in \mathcal{M}_p$, then for $z < 0$,

$$\psi_\mu(z) = \int_{[0,\infty)} \frac{zt}{1-zt}\mu(dt) = \sum_{k=1}^{p} m_k(\mu) z^p + z^p \int_{[0,\infty)} \frac{zt^{p+1}}{1-zt}\mu(dt).$$

- Using fairly standard theory (Bingham, C. M. Goldie, and J. L. Teugels 1989) one can characterize behavior of the remainder term as $z \to 0^-$ when $\bar{\mu}$ is regularly varying.
- We have to understand how this translates to behavior of ψ_μ^{-1}.
- If $\mu \in \mathcal{M}_p$, then we have

$$S_\mu(z) := \frac{z+1}{z}\psi_\mu^{-1}(z) = \sum_{n=0}^{p-1} s_n z^n + r(z), \qquad z \in (\mu\{0\} - 1, 0),$$

where $(s_n)_n$ are real numbers and each s_k is a rational function of moments of μ up to $k+1$. Moreover, $r(z) = o(z^{p-1})$ as $z \to 0^-$.

Idea behind 2

- If $\mu \in \mathcal{M}_p$, then we have with $r(z) = o(z^{p-1})$

$$S_\mu(z) = \sum_{n=0}^{p-1} s_n z^n + r(z), \qquad z \in (\mu\{0\} - 1, 0),$$

  e.g. if $p = 2$, then

$$S_\mu(z) = \frac{1}{m_1(\mu)} - \frac{m_2(\mu) - m_1(\mu)^2}{m_1(\mu)^3} z + r(z).$$

- Regular variation of $\bar{\mu}$ can be described in terms of behavior of r as $z \to 0^-$.
- We work with pth derivative, $r^{(p)} = S_\mu^{(p)}$.
- If $m_1(\mu) = \infty$ ($p = 0$), the behavior of S-transform is very different from the case $m_1(\mu) < \infty$ ($p \in \mathbb{N}$).

Moments of regularly varying measures

From now on, we assume that function L **is slowly varying**.

Recall that

$$\mathcal{M}_p = \{\mu \in \mathcal{M}_+ : m_p(\mu) < \infty \text{ and } m_{p+1}(\mu) = \infty\}.$$

Remark

Assume that $\mu \in \mathcal{M}_+$ is such that $\bar{\mu}(t) = \frac{L(t)}{t^\alpha}$.

Then, μ belongs to $\mathcal{M}_p$, $p \in \mathbb{N}$, if and only if one of the following conditions is satisfied

1. $\alpha \in (p, p+1)$;
2. $\alpha = p$ *and* $\int_1^\infty L(t)/t\, dt < \infty$;
3. $\alpha = p+1$ *and* $\int_1^\infty L(t)/t\, dt = \infty$.

Measure μ belongs to $\mathcal{M}_0$ if and only if one of the following conditions is satisfied

1. $\alpha \in [0, 1)$;
2. $\alpha = 1$ *and* $\int_1^\infty L(t)/t\, dt = \infty$.

Case $\alpha = 0$, $m_1(\mu) = \infty$

Theorem ($\alpha = 0$)
If $\bar{\mu}(x) \sim L(x)$, then the inverse function of

$$x \mapsto 1/S_\mu(-1/x) \tag{1}$$

is slowly varying. Conversely, if the inverse of (1) is slowly varying, then

$$\bar{\mu}(x) \sim 1/f_\mu^{-1}(x),$$

where $f_\mu(x) := (x-1)/S_\mu(-1/x)$.

Example
If $\mu \in \mathcal{M}_+$ is such that

$$S_\mu(z) = e^{\gamma/z}$$

for $\gamma > 0$, then $1/S_\mu(-1/x) = e^{\gamma x}$ and so

$$\bar{\mu}(x) \sim \gamma/\log(x).$$

Case $\alpha \in (0,1)$, $m_1(\mu) = \infty$

Theorem

Let $\alpha \in (0,1)$, $L \in \mathcal{R}_0$ and let $M \in \mathcal{R}_0$ be (unique up to asymptotic equivalence) function satisfying

$$\lim_{x\to\infty} \frac{M(x)}{L^{1/\alpha}(xM(x))} = 1 \qquad \textit{and} \qquad \lim_{x\to\infty} \frac{M(x/L^{1/\alpha}(x))}{L^{1/\alpha}(x)} = 1.$$

Then, the following are equivalent:

$$\bar{\mu}(x) \sim L(x)/x^{\alpha}$$

$$S_\mu\left(-\frac{1}{x}\right) \sim \left(\frac{\sin(\pi\alpha)}{\pi\alpha}\right)^{1/\alpha} \frac{x^{1-1/\alpha}}{M(x^{1/\alpha})},$$

Function M is called the de Bruijn conjugate to $L^{-1/\alpha}$.

Case $\alpha \in (0,1)$, example

- If
  - μ is $\boxplus$-stable of index $1/(1+s)$ and
  - ν is $\boxplus$-stable of index $1/(1+t)$

  then $\mu \boxtimes \nu$ is $\boxplus$-stable of index $1/(1+s+t)$.
- For $\alpha \in (0,2)$, $\boxplus$ α-stable laws have right tail asymptotically equivalent to $c\, x^{-\alpha}$.
- Our result allows us to generalize this observation: If $\mu, \nu \in \mathcal{M}_+$ are such that

$$\overline{\mu} \in \mathcal{R}_{-1/(1+s)} \qquad \text{and} \qquad \overline{\nu} \in \mathcal{R}_{-1/(1+t)},$$

  then

$$x \mapsto \mu \boxtimes \nu\big((x,\infty)\big) \in \mathcal{R}_{-1/(1+s+t)}.$$

Case $\alpha = 1$, $m_1(\mu) = \infty$

Theorem ($\alpha = 1$)
Assume $\int_1^\infty L(t)/t\,dt = \infty$. Then, the following are equivalent:

$$\bar{\mu}(x) \sim L(x)/x$$

$$\lim_{x\to\infty} \frac{\frac{1}{S_\mu(-1/(\lambda x))} - \frac{1}{S_\mu(-1/x)}}{L\big((x-1)/S_\mu(-1/x)\big)} = \log(\lambda), \qquad \lambda > 0.$$

Example
We have $\bar{\mu}(x) \sim 1/x$ if and only if

$$1/S_\mu\left(-\frac{1}{x}\right) = \log(x) + c + \int_0^x o(1)\frac{dt}{t}.$$

Case $\alpha \geq 1$, $m_1(\mu) < \infty$

Theorem

Let $p \in \mathbb{N}$, $\alpha \in [p, p+1]$ and $\mu \in \mathcal{M}_+$.
(i) If

(a) $\alpha \in (p, p+1)$, *or*

(b) $\alpha = p$ *and* $\int_1^\infty L(t)/t\, dt < \infty$,

then $\bar{\mu}(x) \sim \frac{L(x)}{x^\alpha}$ *is equivalent to*

$$S_\mu^{(p)}\left(-\frac{1}{x}\right) \sim -\frac{\Gamma(\alpha+1)\Gamma(p+1-\alpha)}{m_1(\mu)^{\alpha+1}} x^{p+1-\alpha} L(x).$$

(ii) If $\alpha = p+1$ *and* $\int_1^\infty L(t)/t\, dt = \infty$, *then* $\bar{\mu}(x) \sim \frac{L(x)}{x^\alpha}$ *is equivalent to*

$$S_\mu^{(p)}\left(-\frac{1}{x}\right) = -\frac{(p+1)!}{m_1(\mu)^{p+2}} \int_0^x (1+o(1)) \frac{L(t)}{t} dt + c_0.$$

Other tails

Tails at 0^+

- For $\mu \in \mathcal{M}_+$ with $\mu(\{0\}) = 0$ define $\mu^{-1} \in \mathcal{M}_+$ by

$$\mu^{-1}\big((x, \infty)\big) = \mu\big([0, x^{-1})\big).$$

- Using the fact that S_μ and $S_{\mu^{-1}}$ are related by

$$S_{\mu^{-1}}(z) = \frac{1}{S_\mu(-1-z)}, \qquad z \in (-1, 0),$$

we can characterize behavior of $S_\mu(z)$ as $z \to -1^+$ for

$$\mu\left(\left[0, \frac{1}{x}\right]\right) = \frac{L(x)}{x^\alpha}.$$

Example

Haagerup and Möller (2013) considered a family of probability measures $(\mu_{\alpha,\beta} \colon \alpha, \beta \geq 0)$ defined by their S-transforms

$$S_{\mu_{\alpha,\beta}}(z) = \frac{(-z)^\beta}{(1+z)^\alpha}.$$

We have

$$S_{\mu_{\alpha,\beta}}\left(-\frac{1}{x}\right) = \frac{x^{\alpha-\beta}}{(x-1)^\alpha} \sim x^{-\beta}$$

and

$$S_{\mu_{\alpha,\beta}}\left(-1+\frac{1}{x}\right) = x^\alpha \left(1-\frac{1}{x}\right)^\beta \sim x^\alpha.$$

If $\beta > 0$, then

$$\lim_{x\to+\infty} x^{1/(\beta+1)}\bar{\mu}_{\alpha,\beta}(x) = \frac{\sin(\pi\frac{1}{\beta+1})}{\pi\frac{1}{\beta+1}}.$$

If $\alpha > 0$, then

$$\lim_{x\to+\infty} x^{1/(\alpha+1)}\mu_{\alpha,\beta}\big([0,x^{-1})\big) = \frac{\sin(\pi\frac{1}{\alpha+1})}{\pi\frac{1}{\alpha+1}}.$$

Symmetric measures

Arizmendi and Perez-Abreu (2009) showed that one can define the S-transform for arbitrary symmetric probability measure μ. Denote by μ^2 the measure obtained from μ via the push-forward $x \to x^2$. Then

$$S_\mu(z)^2 = \frac{1+z}{z} S_{\mu^2}(z).$$

Moreover the tails of μ and μ^2 are connected via

$$\mu((x, +\infty)) = 1/2\mu^2((x^2, +\infty)).$$

Hence one can use previous results to describe tails of symmetric measures in terms of their S-transform.

Sketch of a proof

Sketch of the proof, $m_1(\mu) < \infty$

Theorem
Let $\alpha \in (p, p+1)$ for $p \in \mathbb{N}$. We have $\bar{\mu}(x) \sim L(x)/x^\alpha$ if and only if

$$S_\mu^{(p)}\left(-\frac{1}{x}\right) \sim -\frac{\Gamma(\alpha+1)\Gamma(p+1-\alpha)}{m_1(\mu)^{\alpha+1}} x^{p+1-\alpha} L(x).$$

1. Assume that

$$\bar{\mu}(x) \sim \frac{L(x)}{x^\alpha}. \tag{2}$$

2. By classical **Abelian-Tauberian** theorems we have that (2) is equivalent to

$$\int_{[0,\infty)} \frac{t^{p+1}}{(t+x)^{p+2}} \mu(dt) \sim \frac{\alpha\Gamma(\alpha+1)\Gamma(p+1-\alpha)}{(p+1)!} \frac{L(x)}{x^{\alpha+1}}.$$

Sketch of the proof

1. But

$$\psi_\mu^{(p+1)}\left(-\frac{1}{x}\right) = (p+1)!\, x^{p+2} \int_{[0,\infty)} \frac{t^{p+1}}{(t+x)^{p+2}} \mu(dt), \tag{3}$$

thus we know the behavior of $x \mapsto \psi_\mu^{(p+1)}(-1/x)$.

Sketch of the proof

1. Let $\chi_\mu := \psi^{-1}$. By expanding the RHS of $0 = \frac{d^{p+1}}{dz^{p+1}}\psi_\mu\left(\chi_\mu(z)\right)$ we obtain
$$\psi_\mu^{(p+1)}(\chi_\mu(z))\chi_\mu'(z)^{p+1} + \psi_\mu'(\chi_\mu(z))\chi_\mu^{(p+1)}(z) = Q(z) \tag{4}$$
where Q on the right hand side is a polynomial in $\psi_\mu^{(l)}(\chi_\mu(z))$ and $\chi_\mu^{(l)}(z)$ for $l = 1, \ldots, p$.
2. Since $\psi_\mu'(\chi_\mu(z)) = 1/\chi_\mu'(z)$, we have as $z \to 0^-$,
$$\chi_\mu^{(p+1)}(z) \sim -\psi_\mu^{(p+1)}(\chi_\mu(z))\chi_\mu'(z)^{p+2} \sim -\frac{\psi_\mu^{(p+1)}\left(\frac{z}{m_1(\mu)}\right)}{m_1(\mu)^{p+2}}.$$
3. By definition we have $S_\mu(z) = \chi_\mu(z) + \frac{1}{z}\chi_\mu(z)$. Thus,
$$S_\mu^{(p)}(z) = \chi_\mu^{(p)}(z) + \frac{d^p}{dz^p}\frac{1}{z}\chi_\mu(z).$$
4. It can be shown (induction and integration by parts) that
$$\frac{d^p}{dz^p}\frac{1}{z}\chi_\mu(z) = \int_z^0 \frac{(-t)^p}{(-z)^{p+1}}\chi_\mu^{(p+1)}(t)dt.$$

Sketch of the proof

$$S_\mu^{(p)}(z) = \chi_\mu^{(p)}(z) + \int_z^0 \frac{(-t)^p}{(-z)^{p+1}} \chi_\mu^{(p+1)}(t)dt, \qquad z \in (\delta - 1, 0).$$

1. This implies that

$$S_\mu^{(p)}\left(-\frac{1}{x}\right) = \chi_\mu^{(p)}\left(-\frac{1}{x}\right) + k_p \overset{M}{*} \tilde{\chi}_{p+1}(x),$$

   where

$$k_p \overset{M}{*} \tilde{\chi}_{p+1}(x) := \int_0^\infty k_p\left(\frac{x}{t}\right)\tilde{\chi}_{p+1}(t)\frac{dt}{t},$$

$$\tilde{\chi}_{p+1}(x) := \chi_\mu^{(p+1)}(-1/x) \quad \text{and}$$

$$k_p(x) := x^{p+1} I_{[0,1]}(x).$$

2. Since $m_p(\mu) < \infty$, we know that $\chi_\mu^{(p)}(z) \to c$ as $z \to 0^-$.

Abelian and Tauberian Theorems

- Abelian theorems allow us to find asymptotics of $k \overset{M}{*} f(x)$ when f is regularly varying with index ρ, then (for nice kernel k we have)

$$\frac{k \overset{M}{*} f(x)}{f(x)} = \int_0^\infty \frac{f\left(\frac{x}{t}\right)}{f(x)} k(t) \frac{dt}{t} \to \int_0^\infty t^{-\rho} k(t) \frac{dt}{t}.$$

- Tauberian theorems give us the converse implication. We assume the regular variation of $k \overset{M}{*} f$ and ask whether this implies the regular variation of f alone. Usually, such converses are true under additional assumptions which are known as 'Tauberian conditions'.
- These 'Tauberian conditions' ensure 'slow oscillations' of f. In particular, 'f is monotone' is one of such conditions.
- In our problem we have
  - $\tilde{\chi}_{p+1}(x) = \chi_\mu^{(p+1)}(-1/x)$ is regularly varying,
  - $S_\mu^{(p)}(-1/x) \sim k_p \overset{M}{*} \tilde{\chi}_{p+1}(x)$.

Thank you!