Almost commuting matrices and stability for product groups

Adrian Ioana

UCSD

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von Neumann (1942): $\exists$ contractions $A_n \in \mathbb{M}_{k_n}(\mathbb{C})$ such that if contractions $B_n \in \mathbb{M}_{k_n}(\mathbb{C})$ satisfy (1), then $\|B_n - \tau(B_n)1\|_2 \to 0$.

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Definition

A sequence $\varphi_n : \Gamma \to \mathrm{U}(d_n)$, for $d_n \in \mathbb{N}$, is an aysmptotic homomorphism if $\|\varphi_n(gh) - \varphi_n(g)\varphi_n(h)\|_2 \to 0$, $\forall g, h \in \Gamma$.

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Γ is HS-stable if for any asymp. homomorphism $\varphi_n : \Gamma \to \mathrm{U}(d_n)$, there are homomorphisms $\pi_n : \Gamma \to \mathrm{U}(d_n)$ such that $\|\varphi_n(g) - \pi_n(g)\|_2 \to 0$, $\forall g \in \Gamma$.

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Becker-Lubotzky (2018): residually finite property (T) groups (e.g., $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$) are not HS-stable.

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4. There is a $*$-homomorphism $\varphi : \mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2) \to \prod_\omega \mathbb{M}_{d_n}(\mathbb{C})$ that does not lift to a $*$-homomorphism $\widetilde{\varphi} : \mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2) \to \ell^\infty(\mathbb{N}, \mathbb{M}_{d_n}(\mathbb{C}))$.

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   Open problem: prove that φ does not admit an **u.c.p.** lift $\widetilde{\varphi}$.
   If no u.c.p. lift exists $\Rightarrow \mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ does not have Kirchberg's local lifting property (open problem going back to **Ozawa, 2003**).

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Proof of Theorem A. Theorem B $\Rightarrow \exists\ U_1, U_2, V_1, V_2 \in \mathcal{U}(\prod_\omega \mathbb{M}_{d_n}(\mathbb{C}))$

1. $[U_i, V_j] = 0$, for all $1 \le i, j \le 2$, but
2. $\nexists$ commuting $*$-subalgebras $P_n, Q_n \subset \mathbb{M}_{d_n}(\mathbb{C})$ such that $U_i \in \prod_\omega P_n, V_i \in \prod_\omega Q_n$, for all $1 \le i \le 2$.

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2. there are no $A_n', B_n' \in \mathbb{M}_{d_n}(\mathbb{C})$ with $A_n'B_n' = B_n'A_n'$, $A_n'B_n'^* = B_n'^*A_n'$, for every n, such that $\|A_n' - A_n\|_2 + \|B_n' - B_n\|_2 \to 0$.

Proof of Theorem A. Theorem B $\Rightarrow \exists\ U_1, U_2, V_1, V_2 \in \mathcal{U}(\prod_\omega \mathbb{M}_{d_n}(\mathbb{C}))$

1. $[U_i, V_j] = 0$, for all $1 \leq i, j \leq 2$, but
2. $\nexists$ commuting $*$-subalgebras $P_n, Q_n \subset \mathbb{M}_{d_n}(\mathbb{C})$ such that $U_i \in \prod_\omega P_n$, $V_i \in \prod_\omega Q_n$, for all $1 \leq i \leq 2$.

Let $f : \mathbb{T} \to [-\frac{1}{2}, \frac{1}{2}]$ be a Borel function s.t. $\exp(2\pi i f(z)) = z, \forall z \in \mathbb{T}$.
Let $h_i = f(U_i)$ and $k_i = f(V_i)$. Let $A = h_1 + ih_2$ and $B = k_1 + ik_2$.

From group stability to almost commuting matrices

Theorem B. $\mathbb{F}_2 \times \mathbb{F}_2$ is not HS-stable.

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Since $W^*(A) = W^*(U_1, U_2)$ and $W^*(B) = W^*(V_1, V_2)$, $\nexists$ commuting $*$-subalgebras $P_n, Q_n \subset \mathbb{M}_{d_n}(\mathbb{C})$ such that $A \in \prod_\omega P_n$ and $B \in \prod_\omega Q_n$.

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Since $W^*(A) = W^*(U_1, U_2)$ and $W^*(B) = W^*(V_1, V_2)$, $\nexists$ commuting $*$-subalgebras $P_n, Q_n \subset \mathbb{M}_{d_n}(\mathbb{C})$ such that $A \in \prod_\omega P_n$ and $B \in \prod_\omega Q_n$.

Since $[A, B] = [A, B^*] = 0$, representing $A = (A_n)$, $B = (B_n)$ and passing to a subsequence gives the conclusion.

First part of the proof of Theorem B, I

Rest of the talk: outline of the proof of Theorem B. It uses f.d. versions of Popa's deformations and spectral gap, and small perturbation results.

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Remark. $\mathbb{F}_k \times \mathbb{F}_m$ is HS-stable iff $\forall \varepsilon > 0$, $\exists \delta > 0$ such that:
$\forall U_1, .., U_k, V_1, .., V_m \in \mathrm{U}(d)$ such that $\|[U_i, V_j]\|_2 < \delta$, $\forall i, j$,

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Proposition C

Assume that $\mathbb{F}_2 \times \mathbb{F}_2$ is HS-stable. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that:

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In other words, $\mathbb{F}_k \times \mathbb{F}_m$ are HS-stable and satisfy an averaged version of HS-stability, uniformly over $k, m \in \mathbb{N}$.

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In other words, $\mathbb{F}_k \times \mathbb{F}_m$ are HS-stable and satisfy an averaged version of HS-stability, uniformly over $k, m \in \mathbb{N}$. (δ depends on ε but not on k, m).

First part of the proof of Theorem B, II

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To prove Proposition C, we use pairs of unitaries with spectral gap and a small perturbation lemma. These are combined via a matrix trick.

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Lemma (pairs of unitaries with spectral gap)

There is a universal $\kappa > 0$ such that $\forall n \in \mathbb{N}$, $\exists X_1, X_2 \in \mathrm{U}(n)$ satisfying

$$\|A - \tau(A)1\|_2 \leq \kappa(\|[X_1, A]\|_2 + \|[X_2, A]\|_2), \forall A \in \mathbb{M}_n(\mathbb{C}).$$

First part of the proof of Theorem B, II

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First part of the proof of Theorem B, II

Proposition C

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To outline the proof of Proposition C, assume that $\mathbb{F}_3 \times \mathbb{F}_3$ is HS-stable.

First part of the proof of Theorem B, II

Proposition C

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To outline the proof of Proposition C, assume that $\mathbb{F}_3 \times \mathbb{F}_3$ is HS-stable. Let $U_1, .., U_k, V_1, .., V_m \in \mathrm{U}(d)$ such that $\|[U_i, V_j]\|_2 \approx 0, \forall i, j$.

First part of the proof of Theorem B, III

Let $X_1, X_2 \in \mathsf{U}(k)$, Y_1, $Y_2 \in \mathsf{U}(m)$ be pairs of unitaries with spectral gap.

First part of the proof of Theorem B, III

Let $X_1, X_2 \in \mathsf{U}(k)$, Y_1, $Y_2 \in \mathsf{U}(m)$ be pairs of unitaries with spectral gap. Define Z_1, Z_2, Z_3, T_1, T_2, $T_3 \in \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_d(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C}) = \mathbb{M}_{kdm}(\mathbb{C})$ by

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Then $\|[Z_i, T_j]\|_2 \approx 0$, $\forall 1 \leq i, j \leq 3$.

First part of the proof of Theorem B, III

Let $X_1, X_2 \in \mathrm{U}(k)$, $Y_1, Y_2 \in \mathrm{U}(m)$ be pairs of unitaries with spectral gap. Define $Z_1, Z_2, Z_3, T_1, T_2, T_3 \in \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_d(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C}) = \mathbb{M}_{kdm}(\mathbb{C})$ by
$Z_1 = X_1 \otimes 1 \otimes 1, \;\; Z_2 = X_2 \otimes 1 \otimes 1, \;\; Z_3 = \sum_{i=1}^k e_{i,i} \otimes U_i \otimes 1,$
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Then $\|[Z_i, T_j]\|_2 \approx 0$, $\forall 1 \leq i, j \leq 3$. $\mathbb{F}_3 \times \mathbb{F}_3$ HS-stable $\Rightarrow \exists$ $*$-subalgebra $P \subset \mathbb{M}_{kdm}(\mathbb{C})$ such that $Z_1, Z_2, Z_3 \in_\sim P$ and $T_1, T_2, T_3 \in_\sim Q = P'$.

First part of the proof of Theorem B, III

Let $X_1, X_2 \in \mathrm{U}(k)$, $Y_1, Y_2 \in \mathrm{U}(m)$ be pairs of unitaries with spectral gap.
Define $Z_1, Z_2, Z_3, T_1, T_2, T_3 \in \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_d(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C}) = \mathbb{M}_{kdm}(\mathbb{C})$ by
$Z_1 = X_1 \otimes 1 \otimes 1, \quad Z_2 = X_2 \otimes 1 \otimes 1, \quad Z_3 = \sum_{i=1}^{k} e_{i,i} \otimes U_i \otimes 1,$
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Then $\|[Z_i, T_j]\|_2 \approx 0$, $\forall 1 \leq i, j \leq 3$. $\mathbb{F}_3 \times \mathbb{F}_3$ HS-stable $\Rightarrow \exists$ $*$-subalgebra $P \subset \mathbb{M}_{kdm}(\mathbb{C})$ such that $Z_1, Z_2, Z_3 \in_\sim P$ and $T_1, T_2, T_3 \in_\sim Q = P'$.

Definition. Let $A, B \subset \mathbb{M}_n(\mathbb{C})$ be $*$-subalgebras.
We write $A \subset_\varepsilon B$ if $\forall x \in A, \|x\| \leq 1, \exists y \in B$ such that $\|x - y\|_2 \leq \varepsilon$.
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P almost commutes with T_1, T_2 $\Rightarrow P \subset_\sim \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_d(\mathbb{C}) \otimes 1$.

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If $P \subset_\sim \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_d(\mathbb{C}) \otimes 1$ and $Q = P' \subset_\sim 1 \otimes \mathbb{M}_d(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C})$, then

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Second part of the proof of Theorem B, I

Construction of almost commuting sets: let $n \in \mathbb{N}$ and $t > 0$.

1. Let $M_n = \otimes_{k=1}^{n} \mathbb{M}_2(\mathbb{C})$ and $A_n = \otimes_{k=1}^{n} \mathbb{C}^2$.

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Claim. $\mathcal{U}_n = \{X_{n,i} \otimes 1\}_{i=1}^n$ and $\mathcal{V}_n = G_n \cup \theta_{t,n}(G_n)$ almost commute: $\|[U, V]\|_2 \leq 4t$, for every $U \in \mathcal{U}_n$ and $V \in \mathcal{V}_n$.

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Second part of the proof of Theorem B, II

If $\mathbb{F}_2 \times \mathbb{F}_2$ is assumed HS-stable, then applying Proposition C to the almost commuting sets $\mathcal{U}_n$ and $\mathcal{V}_n$ implies that $\exists$ $*$-subalgebra $C_n \subset A_n = \otimes_{k=1}^{n} \mathbb{C}^2$ such that

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- If C_n satisfies (1), then $\dim(C_n) \leq P(n)$, for a polynomial $P(n)$.

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  Specifically, if $\frac{1}{n}\sum_{i=1}^{n} \|X_{n,i} - E_{C_n}(X_{n,i})\|_2^2 \leq \varepsilon$, then

$$\dim(C_n) \geq 2^{n - \mathrm{H}(4\varepsilon)n - 3},$$

  where $\mathrm{H}(\delta) = -\delta \log_2(\delta) - (1-\delta)\log_2(1-\delta)$.