Heat flow, random matrices, and random polynomials

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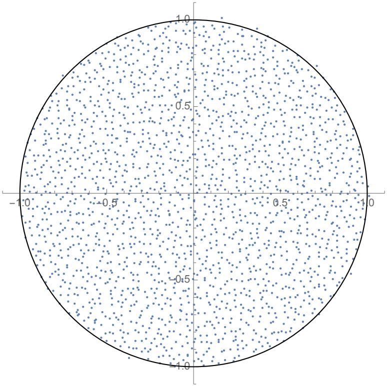
INTRODUCTION:
Random matrices and heat flow

Random matrices: Circular law

- **Ginibre ensemble**: $N \times N$ matrix Z with indep. entries
- Each entry complex Gaussian of mean 0, variance $1/N$
- When N is large, eigenvalues will be approx. uniform on unit disk:



Random matrices: Circular law

- Define (random) **empirical eigenvalue measure** of Z as

$$\mu^N = \frac{1}{N}\sum_{j=1}^{N}\delta_{\lambda_j}$$

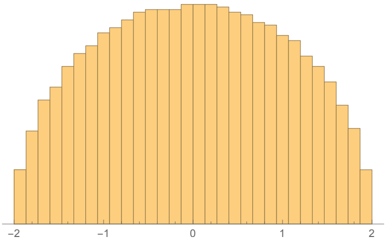
where $\{\lambda_1, \ldots, \lambda_N\}$ are eigenvalues of Z

Theorem

The random probability measure μ^N converges weakly almost surely to the uniform probability measure on unit disk

Random matrices: Semicircular law

- **Gaussian unitary ensemble**: $N \times N$ Hermitian matrix with indep. entries on and above diagonal, with $X_{kj} = \overline{X_{jk}}$
- Entries on diagonal are real Gaussian with mean 0, variance $1/N$
- Entries off diagonal are complex Gaussian with mean 0, variance $1/N$
- Eigenvalues approx. semicircular shape on $[-2, 2]$:



Heat flow on polynomials: definition

- Heat operator on polynomial p of degree N:

$$\exp\left\{\frac{\tau}{2N}\frac{d^2}{dz^2}\right\}p(z) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\tau}{2N}\right)^k\left(\frac{d}{dz}\right)^{2k}p(z), \quad \tau \in \mathbb{C},$$

- Series terminates for all $\tau \in \mathbb{C}$
- Will always put factor of $N = \deg(p)$ in denominator
- This is natural scaling of time variable

Heat flow on polynomials: evolution of zeros

- Zeros $z_j(\tau)$ satisfy

$$\frac{dz_j}{d\tau} = -\frac{1}{N}\sum_{k:\, k\neq j} \frac{1}{z_j(\tau) - z_k(\tau)}$$

- Also

$$\frac{d^2 z_j}{d\tau^2} = -\frac{2}{N^2}\sum_{k:\, k\neq j} \frac{1}{(z_j(\tau) - z_k(\tau))^3}$$

- Formula for second deriv. is **rational Calogero–Moser system**

Backward heat flow and random matrices

- Take Hermitian random matrix X^N with

$$\text{e.v. distribution } \to \mu$$

- Let p^N be char. poly. of X^N
- Apply **backward** heat op. $(\tau = -t)$ to get poly. p_t^N
- Pólya–Benz theorem [1934]: roots will remain real

Theorem (Kabluchko)

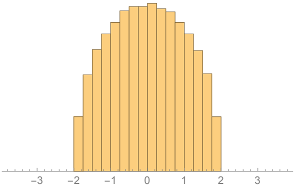
Empirical measure of zeros of p_t^N approach free additive convolution:

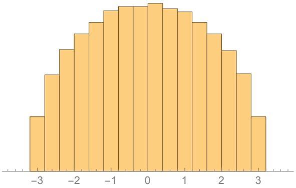
$$\mu \boxplus (\textit{semi. circ. measure on } [-2\sqrt{t}, 2\sqrt{t}])$$

Result of Kabluchko using "finite free convolution" of Marcus, Spielman, Srivastava

Backward heat flow and random matrices

- **Free convolution**: computes limiting e.v. distribution of sums of indep. Hermitian random matrices
- Hence: zeros of p_t^N resemble e.v. of $X^N + \sqrt{t}\,\mathrm{GUE}$
- **Random matrix interpretation** to backward heat op on polynomials with real roots: like adding a GUE!
- **Example**: backward heat flow on char. poly. GUE gives semicirc. distrib. on $[-2\sqrt{1+t}, 2\sqrt{1+t}]$





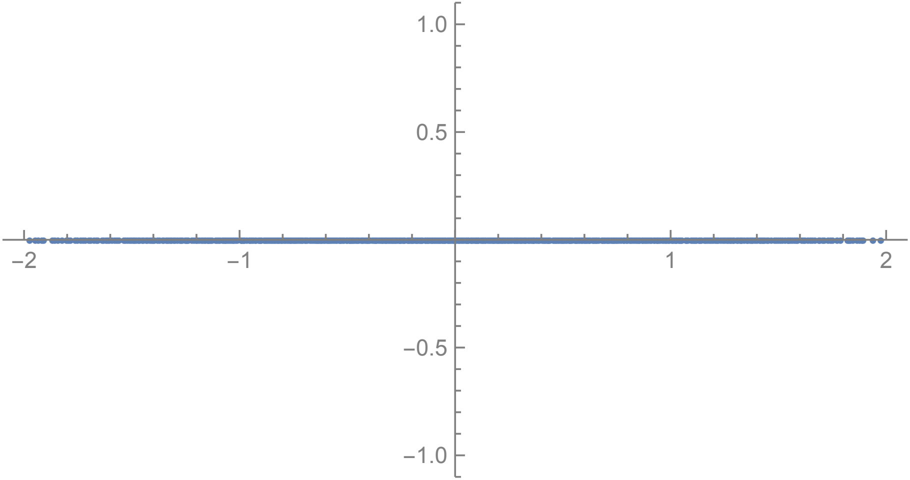
Forward heat flow and random matrices

Question

What happens if we apply **forward** *heat operator ($\tau = t$) to characteristic polynomial of GUE? Can we just replace t by $-t$ in preceding result (shrinking semicircle)?*

- Let's see!

Forward heat flow and random matrices



Forward heat flow and random matrices

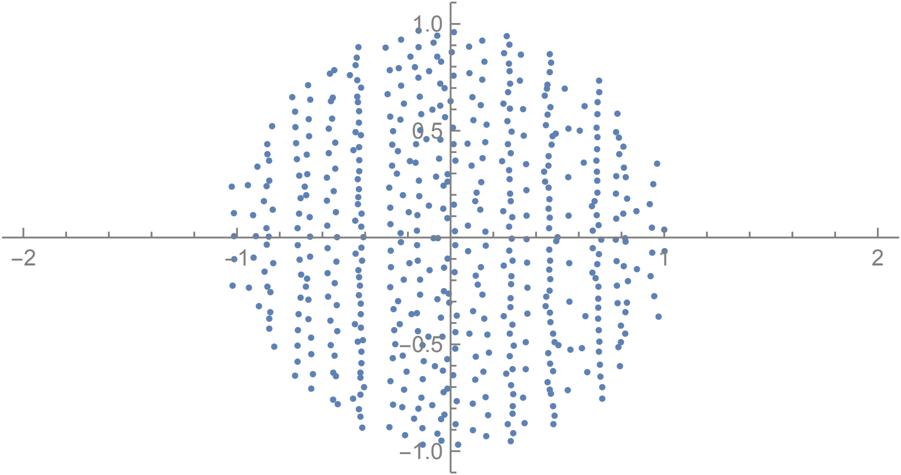
- Apply *forward* heat op. ($\tau = t$) to char. poly. of GUE
- Zeros become **complex** even for small t if N is large

Conjecture (Hall–H0, '22)

For $0 < t < 2$, get uniform distrib. on ellipse with semi-axes $2 - t$ and t. E.g., with $t = 1$, get uniform distribution on disk: semicircular becomes circular!

Example: Semicircular to circular

- Forward heat op. for time $\tau = 1$ starting from char. poly. GUE
- Approx. uniform on unit disk—but not same distrib. as e.v. of Ginibre



Why does result for backward heat operator not extend?

- Method of Marcus–Spielman–Srivastava uses **expected char. poly.**
- *Expectation value* of (forward heat op.)(char. poly. GUE) is scaled Hermite polynomial
- Zeros of expected polynomial *will* have shrinking semicircular distribution
- But this does not tell you about zeros without expectation value—unless zeros are real
- Later: use expectation value of **absolute value squared** of char. poly.

Heat flow and random matrix theory

Goal

Identify examples in which applying heat operator to characteristic polynomial of **one** *random matrix model gives new polynomial whose zeros resemble the eigenvalues of a* **second** *random matrix model.*

Goal

Develop **general theory** *of how zeros of polynomials evolve under heat flow, apart from connections to random matrix theory.*

PART 2

MODEL DEFORMATION PHENOMENON

The relationship between circular and semicircular laws

- Twice the real part of the eigenvalues in the circular law has the same bulk distribution as the eigenvalues in the semicircular law
- Trivial from the formulas but: **why** is it true?
- Why are real parts of eigenvalues in one model related to the eigenvalues in different model?

Circular-semicircular Challenge

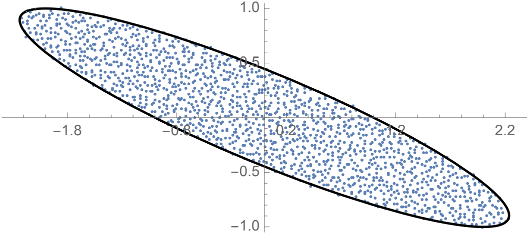
Explain the **relationship** *between limiting e.v. distributions of Ginibre ensemble and GUE without using the circular and semicircular laws.*

Generalizing: Hermitian plus elliptic model

- RMT: Let X and Y be independent GUE's, set

$$Z = e^{i\theta}(aX + ibY)$$

- Free prob.: take X, Y freely indep. semicircular elements
- Limiting e.v. distribution/Brown measure is uniform on ellipse:



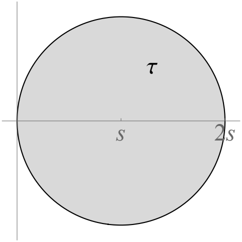
Parameters for elliptic model

- Use parameters $s \in \mathbb{R}$ and $\tau \in \mathbb{C}$

$$s = \mathbb{E}\left\{\frac{1}{N}\mathrm{trace}(Z^* Z)\right\} \quad \textbf{(variance)}$$

$$\tau = \mathbb{E}\left\{\frac{1}{N}\mathrm{trace}(Z^* Z)\right\} - \mathbb{E}\left\{\frac{1}{N}\mathrm{trace}(Z^2)\right\}$$

- **Special cases**: $\tau = 0$ is Hermitian, $\tau = s$ is circular
- From Cauchy–Schwarz: $|\tau - s| \leq s$



Elliptic plus Hermitian model

- Label elliptic element as $Z_{s,\tau}$; consider

$$X_0 + Z_{s,\tau}$$

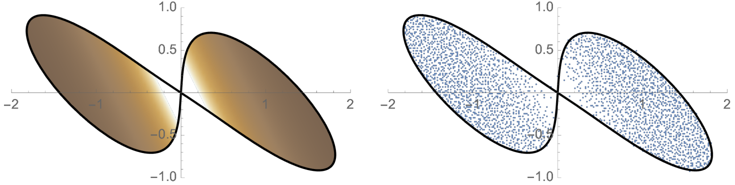
  where X_0 is Hermitian, indep. of $Z_{s,\tau}$
- Let μ be limiting e.v. distribution of X_0
- Additive case of work of Hall–Ho [2021]; results of Zhong [2021]

Elliptic plus Hermitian model

Theorem (Hall–Ho; Zhong)

Limiting e.v. distribution of $\mu_{s,\tau}$ of $X_0 + Z_{s,\tau}$ is supported on explicitly computable domain $\Omega_{s,\tau}$ and density of $\mu_{s.\tau}$ is constant (in $\Omega_{s,\tau}$) in the $i\tau$ direction.

- Example: X_0 is Bernoulli: μ is half sum of δ-measures at ± 1
- $s = 1$, $\tau = 1 + i/2$



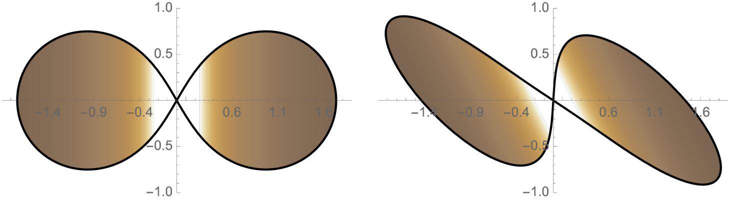
“Model deformation” result: vary τ with s fixed

- Fix s and X_0, take τ_0 and τ

Theorem (Hall–Ho; Zhong)

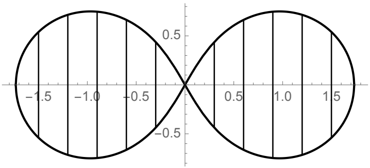
There is a canonical map $\Phi_{s,\tau_0,\tau}$ such that push-forward of Brown measure of $X_0 + Z_{s,\tau_0}$ by $\Phi_{s,\tau_0,\tau}$ equals Brown measure of $X_0 + Z_{s,\tau}$.

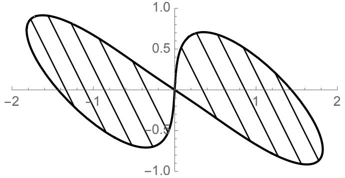
- Bernoulli case with $s = 1$, $\tau_0 = 1$, and $\tau = 1 + i/2$



Pictorial description of map

- Map takes segments in $i\tau_0$-direction to segments in $i\tau$-direction
- $\Phi_{s,\tau_0,\tau}(z)$ is **linear in** τ for fixed z
- Bernoulli case with $s = 1$, $\tau_0 = 1$, and $\tau = 1 + i/2$





HOW to vary τ with s fixed?

- Considering $Z_{s,\tau}$ with fixed s and different values of τ
- *Cannot* change τ with s fixed by adding an indep. matrix ΔZ
- s is variance and variances add!
- *Can* decrease variance in Hermitian directions and increase variance in skew-Hermitian directions
- **Fiction**: Add an element of form $X_{-r} + iY_r$, semicircular with variances $-r$ and r
- Makes sense on element of form $W + \tilde{X}_t$, goes to $W + \tilde{X}_{t-r} + iY_r$

Circular–semicircular case

- Take $X_0 = 0$, with $s = 1$
- Take $\tau_0 = 1$ (circular) and $\tau = 0$ (semicircular)
- Map $\Phi_{s,\tau_0,\tau}$ gives circular-to-semicircular map:

$$\Phi_{s,\tau_0,\tau}(z) = 2\,\mathrm{Re}(z)$$

Conclusion

The map $z \mapsto 2\,\mathrm{Re}(z)$ relating circular to semicircular laws is just one special case of a large family of maps with similar results.

PDE Method

- **PDE method** for proving these results

Proposition (Hall–Ho, 2023)

Log potential $S(z, s, \tau)$ of Brown measure $\mu_{s,\tau}$ satisfies a PDE w.r.t. τ with s fixed:

$$\frac{\partial S}{\partial \tau} = \frac{1}{2}\left(\frac{\partial S}{\partial z}\right)^2.$$

- Here S is real-valued, τ and z are complex variables
- Derivatives are complex partial derivatives (Cauchy–Riemann operators)

Multiplicative models

- Gaussian models: *sums* of i.i.d. matrices
- Also consider **products** of i.i.d. matrices close to identity
- Take

$$B_{s,\tau} = \prod_{j=1}^{k} \left(I + \frac{i}{\sqrt{k}} Z_{s,\tau}^{j} - \frac{1}{2k}(s-\tau)I \right), \quad k \gg 1,$$

  where $Z_{s,\tau}^{j}$'s are independent copies of $Z_{s,\tau}$
- Or: solve free SDE driven by elliptic Brownian motion:

$$dB_{s,\tau}(t) = B_{s,\tau}(t) \left(i\, dZ_{s,\tau}(t) - \frac{1}{2}(s-\tau)dt \right)$$

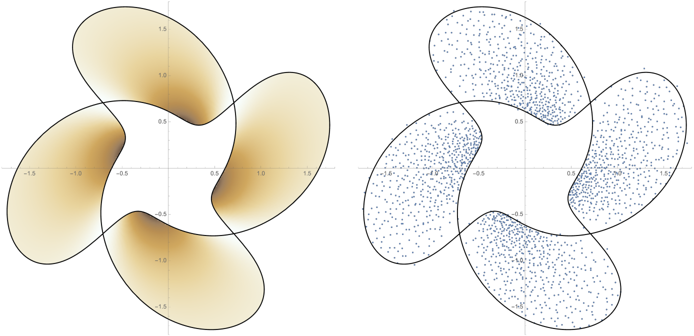
  then set $t = 1$

Multiplicative models

- Then take U unitary, indep. of $B_{s,\tau}$ and consider

$$UB_{s,\tau}$$

- Limiting e.v. distribution of $UB_{s,\tau}$ computed in increasing generality by Driver–Hall–Kemp, Ho–Zhong, Hall–Ho
- **Example**: Law of U at ± 1 and $\pm i$ with $s = 1$, $\tau = 1 + i/2$

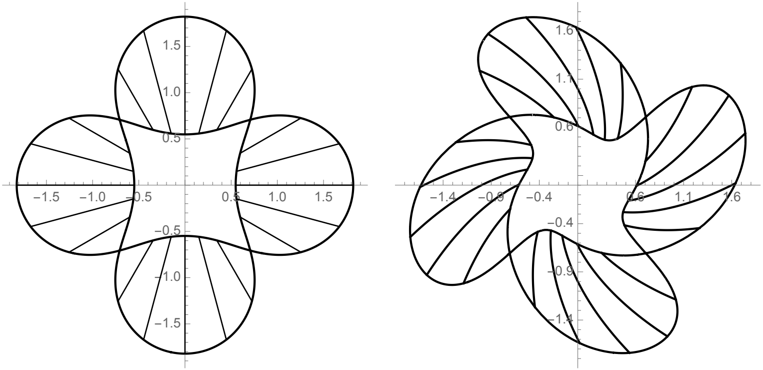


Relating models with different values of τ

Theorem (Hall–Ho, 2023)

"Model deformation" holds in the multiplicative case.

- Fix s and U, relate different values of τ with map
- Example: $s = 1$, $\tau_0 = 1$, $\tau = 1 + i/2$



PART 3

HEAT FLOW CONJECTURE FOR RANDOM MATRICES

Idea of heat flow conjecture

Idea

Transformation between random matrices with different values of τ can be accomplished by **applying heat operator to characteristic polynomial** *of one model.*

Heat flow

- Fix X_0 and s, take τ_0 and τ
- Set p_0^N = char. poly. of $X_0 + Z_{s,\tau_0}$
- Set p^N = char. poly. of $X_0 + Z_{s,\tau}$
- Set

$$q^N(z) = \exp\left\{\frac{\tau - \tau_0}{2N}\frac{\partial^2}{\partial z^2}\right\} p_0^N(z)$$

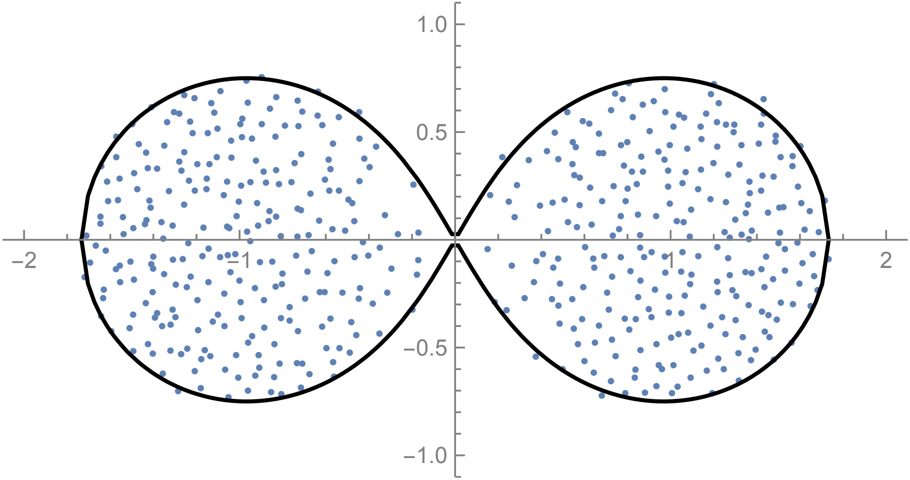
Conjecture (Hall–Ho)

The empirical measure for zeros of q^N converges weakly almost surely to the same limit as for zeros of p^N—namely, limiting e.v. distrib. of $X_0 + Z_{s,\tau}$.

Heat flow

- p^N is char. poly. of random matrix with parameter τ
- q^N: start with char. poly. of random matrix with parameter τ_0, apply heat flow for time $\tau - \tau_0$
- **Conjecture**: zeros of $q^N \approx$ zeros of p^N
- Heat flow for time $\tau - \tau_0$ **changes from** τ_0 **to** τ

Bernoulli case

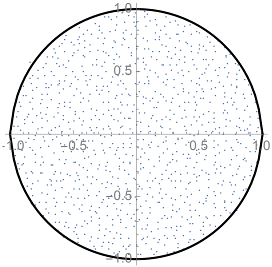


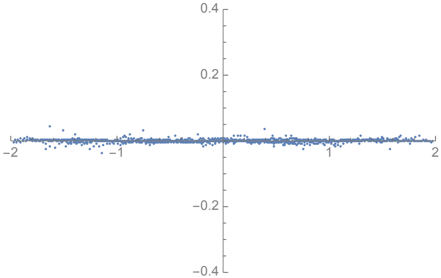
Circular to semicircular case

- Take $X_0 = 0$, $s = 1$
- Take $\tau_0 = 1$ (circular) and $\tau = 0$ (semicircular)
- Roots of

$$q^N := \exp\left\{-\frac{1}{2N}\frac{\partial^2}{\partial z^2}\right\} p_0^N$$

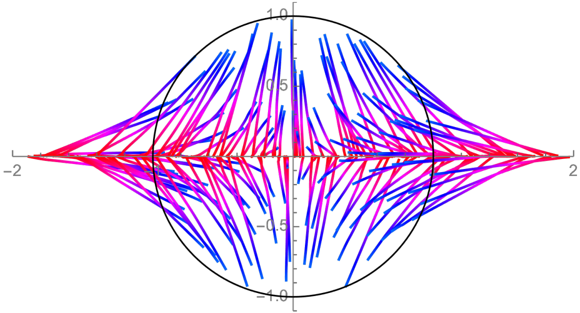
approximate semicircular distribution on $[-2, 2]$





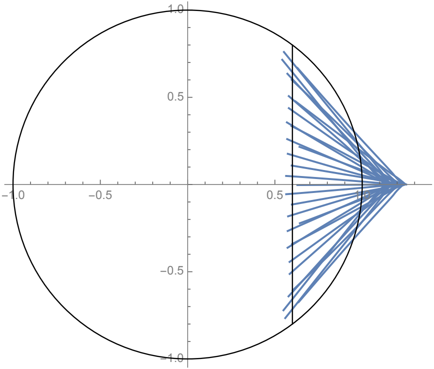
Trajectories in circular to semicircular case

- Put t in exponent with $0 \leq t \leq 1$
- Zeros move in approx. **straight lines** $z \mapsto z + t\bar{z}$
- Motion reflects that $\Phi_{s,\tau_0,\tau}(z)$ is linear in τ for fixed z
- Point starting at z ends close to $2\,\mathrm{Re}(z)$

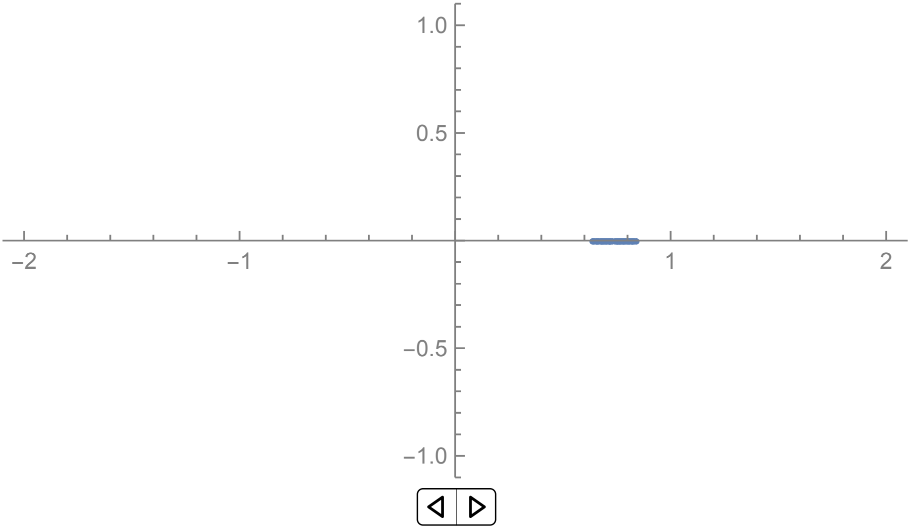


Semicircular to circular case ($\tau_0 = 0$, $\tau = 1$)

- Points move in approx. straight lines
- Velocity in x-direction determined by initial x-value
- Velocity in y-direction random
- Points starting near given x-value end up on vertical line



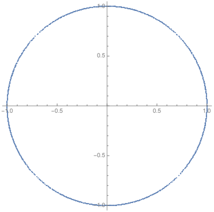
Forward heat flow and random matrices

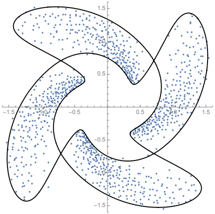


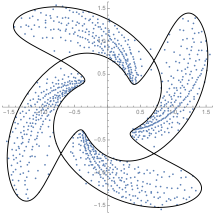
Multiplicative case

- Similar results, with heat operator replaced by

$$\exp\left\{-\frac{\tau-\tau_0}{2N}\left(z^2\frac{\partial^2}{\partial z^2}-(N-2)z\frac{\partial}{\partial z}-N\right)\right\}$$







Part 4

GENERAL HEAT FLOW CONJECTURE

General conjecture

- Let p_0^N be deg.-N polynomials s.t. empirical measure of zeros converges to "nice" measure μ
- Define

$$p^N(\tau, z) = \exp\left\{\frac{\tau}{2N}\frac{d^2}{dz^2}\right\} p_0^N(z)$$

Conjecture

For sufficiently small $|\tau|$, the empirical measure of zeros of $p^N(\tau, z)$ converges to measure μ_τ whose log potential $S(\tau, z)$ satisfies

$$\frac{\partial S}{\partial \tau} = \frac{1}{2}\left(\frac{\partial S}{\partial z}\right)^2.$$

Notes

- S is real-valued but τ and z are complex variables
- $\partial/\partial\tau$ and $\partial/\partial z$ are complex partial deriv. (Cauchy–Riemann ops.)
- Polynomials needn't come from random matrices
- There is multiplicative version of conjecture
- Solution S can degenerate; can't expect C^1 solution for all τ
- Expect **straight-line motion** for small τ:

$$z \mapsto z + \tau \frac{\partial S(0,z)}{\partial z}$$

Formal argument

- Define log potential of zeros $\{z^j(\tau)\}_{j=1}^N$ of $p^N(\tau, z)$

$$S^N(\tau, z) = \frac{1}{N}\sum_{j=1}^{N} \log(|z - z_j(\tau)|^2)$$

Proposition

The log potential of the zeros S^N satisfies the PDE

$$\frac{\partial S^N}{\partial \tau} = \frac{1}{2}\left(\frac{\partial S^N}{\partial z}\right)^2 + \frac{1}{2N}\frac{\partial^2 S^N}{\partial z^2}$$

away from the zeros.

- **But**: this is *not* viscosity approx. to PDE in conjecture

SUPPORTING THE CONJECTURES:
Rigorous results

First Rigorous Result: Second moments of char. poly.

- Fix X_0 and s, take τ_0 and τ
- Set $p_0^N =$ char. poly. of $X_0 + Z_{s,\tau_0}$
- Set $p^N =$ char. poly. of $X_0 + Z_{s,\tau}$
- Set
$$q^N(z) = \exp\left\{\frac{\tau - \tau_0}{2N}\frac{\partial^2}{\partial z^2}\right\} p_0^N(z)$$
- **Goal**: Show p^N and q^N have similar bulk distribution of zeros

First Rigorous Result: Second moments of char. poly.

Theorem (Hall–Ho)

For all $z \in \mathbb{C}$, we have

$$\mathbb{E}\left\{ |q^N(z)|^2 \right\} = \mathbb{E}\left\{ |p^N(z)|^2 \right\}$$

Proof.

Both sides satisfy the PDE

$$\frac{\partial u}{\partial \tau} = \frac{1}{2N} \frac{\partial^2 u}{\partial z^2}$$

with equality at $\tau = \tau_0$. $\square$

Significance of the second moment

- If p is a degree-N polynomial,

$$\text{empirical measure of zeros of } p = \frac{1}{4\pi N}\Delta \log(|p(z)|^2)$$

- Assume **concentration**—$|p^N(z)|^2 \approx \mathbb{E}\{|p^N(z)|^2\}$
- Then can freely insert a expectation:

$$\text{empirical measure of zeros of } p^N \approx \frac{1}{4\pi N}\Delta \log(\mathbb{E}\{|p^N(z)|^2\})$$

- **Conclusion**: Hope to recover zeros of p^N and q^N from second moments—which are equal!

Second rigorous result: Polynomials with independent coefficients

- Work in progress with Ho, Jalowy, and Kabluchko
- Kabluchko and Zaporozhets have analyzed wide class of polynomials with independent coefficients
- Apply (backward) heat operator for time t
- First: extend KZ results by applying heat flow to the polynomials
- Second: Verify that results agree with general heat flow conjecture

Second rigorous result: Polynomials with independent coefficients

Theorem (Hall–Ho–Jalowy–Kabluchko, 2023+)

1. *Log potential S of limiting zero-distribution of heat-evolved KZ polynomials satisfies the PDE*

$$\frac{\partial S}{\partial \tau} = \frac{1}{2}\left(\frac{\partial S}{\partial z}\right)^2$$

2. *The limiting zero-distribution at time τ is push-forward of distribution at time 0 under an explicit* **transport map** T_t

Weyl polynomials: Rigorous circular-to-semicircular result

- Consider Weyl polynomials W_N:

$$W_N(z) = \sum_{j=0}^{N} \xi_j \frac{(\sqrt{N}z)^j}{\sqrt{j!}},$$

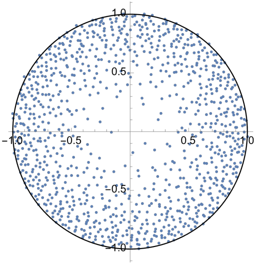
  where $\{\xi_j\}$ are i.i.d. standard complex Gaussians
- Limiting distribution of zeros is uniform on unit disk (circular law)
- Transport map is

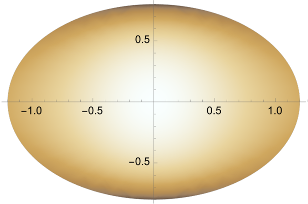
$$T_t(z) = z + t\bar{z}$$

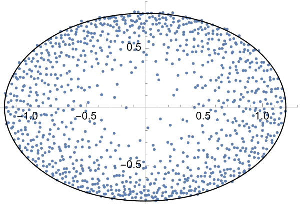
- Limiting distrib. of zeros of heat-evolved poly. is uniform on ellipse for $0 < t < 1$
- Limiting distrib. is semicircular on $[-2, 2]$ for $t = 1$

Littlewood–Offord polynomials

- Start from Littlewood–Offord poly. with $\beta = 1/4$
- Distrib. at $t = 0$ is quadratic on unit disk
- Push forward by *explicit* map T_t of disk to ellipse







Third rigorous result: Gaussian Analytic Function (GAF)

- GAF is "infinite Weyl polynomial" (without factor of $\sqrt{N}$)

Definition

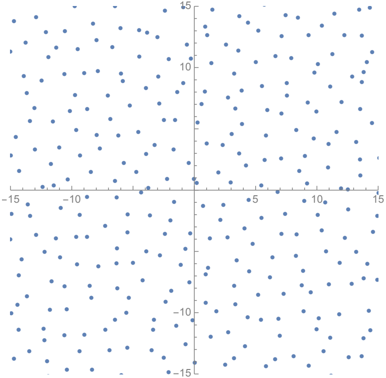
The GAF is the random entire function given by

$$\sum_{j=0}^{\infty} \xi_j \frac{z^j}{\sqrt{j!}}$$

where $\{\xi_n\}$ are i.i.d. standard complex Gaussians.

Zeros of GAF

- Zeros of GAF form an interesting random set of points in the plane
- Zeros are invariant (in distrib.) under rotations and translations



GAF under heat flow

- Makes sense to apply $e^{\frac{\tau}{2}\frac{d^2}{dz^2}}$ to G, if $|\tau| < 1$

Theorem (Hall–Ho–Jalowy–Kabluchko, 2023+)

For all $\tau \in \mathbb{C}$ with $|\tau| < 1$ the function

$$(V_\tau G)(z) := (1 - |\tau|^2)^{1/4} e^{-\frac{\tau}{2}z^2} \left(e^{\frac{\tau}{2}\frac{d^2}{dz^2}} G \right) \left(\sqrt{1 - |\tau|^2}\, z \right)$$

has the same distribution as G.

- Hence: GAF remains invariant in distribution under heat flow, up to some simple transformations

Zeros of GAF under heat flow

- Constant and Gaussian factor don't affect zeros

Corollary

If $z_j(\tau)$ *are zeros of* $e^{\frac{\tau}{2}\frac{d^2}{dz^2}}G$, *then*

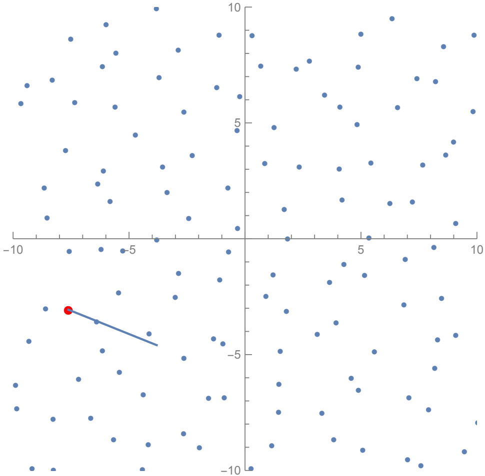
$$\left\{ \frac{z_j(\tau)}{\sqrt{1-|\tau|^2}} \right\}$$

have same distribution as zeros of G *(for* $|\tau| < 1$*).*

- So we understand how zeros of an "infinite-degree random polynomial" transform under heat flow!
- Exact result at level of individual zeros (not just bulk level)

GAF: dynamics of individual zeros

- Zeros $z_j(\tau)$ tend to move along straight lines: $z \mapsto z - \tau\bar{z}$



GAF: dynamics of individual zeros

- Let G^a denote GAF G conditioned to have a zero at $a \in \mathbb{C}$
- Let $z^a(\tau)$ denote the zero of $e^{\frac{\tau}{2}\frac{d^2}{dz^2}} G^a$ that starts at a

Theorem (Hall–Ho–Jalowy–Kabluchko, 2023+)

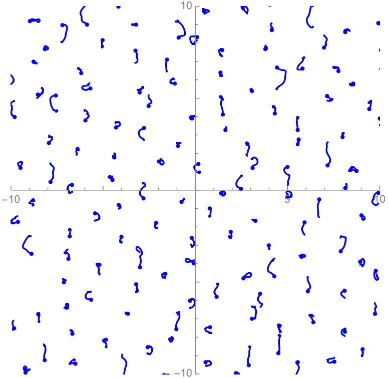
We have the following equality in distribution:

$$z^a(\tau) \stackrel{d}{=} a - \tau\bar{a} + z^0(\tau)$$

- $z^0(\tau)$ is a fixed random variable with distrib. indep. of a
- Result says that zero evolves in straight line, plus "order 1" error

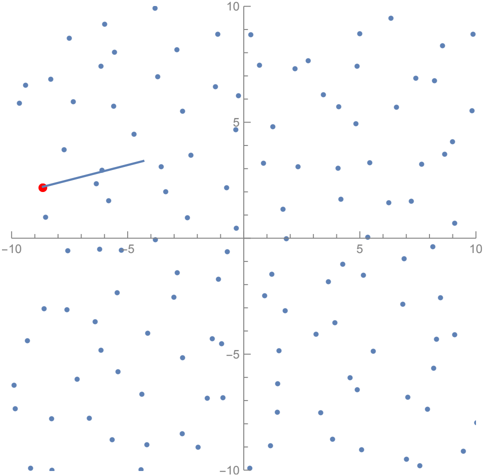
GAF: dynamics of individual zeros

- Plots of zeros with straight-line motion subtracted off
- I.e., plot $z_j(\tau) - (z_j(0) - \tau\overline{z_j(0)})$
- All points then move with same scale



Conclusion

THANK YOU FOR YOUR ATTENTION



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