Towards a classification of multi-faced independences

Malte Gerhold

University of Greifswald

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based on joint work with Takahiro Hasebe and Michaël Ulrich
(arxiv:2111.07649)

Overview

Overview

Non-commutative independence

Fix product operation for linear functionals on $*$-algebras

$$\bigtimes_i {B_i}' \ni (\varphi_i)_i \mapsto \bigodot_i \varphi_i \in \Big(\bigsqcup_i B_i\Big)'$$

Def: $\odot$-independence of random variables $j_i \colon B_i \to \mathcal{A}$

$$\Phi \circ \bigsqcup_i j_i = \bigodot_i (\Phi \circ j_i)$$

joint distribution = product of marginals

uau-products

Def: universal product, Ben Ghorbal & Schürmann '05

$$B_1' \times B_2' \ni (\varphi_1, \varphi_2) \mapsto \varphi_1 \odot \varphi_2 \in (B_1 \sqcup B_2)'$$

product operation (for arbitrary algebras B_1, B_2) which is

- **u**nital in the sense that $0 \odot \varphi = \varphi = \varphi \odot 0$
- **a**ssociative
- **u**niversal in the sense that (for hom's $j_i \colon B_i \to A_i$)

$$(\varphi_1 \odot \varphi_2) \circ (j_1 \sqcup j_2) = (\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2)$$

Examples

tensor, free, monotone, anti-monotone, Boolean

uau-products

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$$(\varphi_1 \odot \varphi_2) \circ (j_1 \sqcup j_2) = (\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2)$$

Theorem: classification (Muraki 2002, 2013)

positivity or *double normalization* $\Longrightarrow$ no other examples

Properties of universal product $\odot$

Def: **positive**

$$\varphi_1, \varphi_2 \text{ rest. states} \implies \varphi_1 \odot \varphi_2 \text{ rest. state}$$

(*restricted state*: unital extension is a state)

Def: **symmetric**

$$\varphi_1 \odot \varphi_2 = \varphi_2 \odot \varphi_1$$

(with identification $A_1 \sqcup A_2 \equiv A_2 \sqcup A_1$)

Def.: **respecting units**

$$\varphi_1, \varphi_2 \text{ unital} \implies \varphi_1 \odot \varphi_2(a(1_1 - 1_2)b) = 0$$

i.e. $\varphi_1 \odot \varphi_2$ well defined on $A_1 \sqcup A_2/(1_1 = 1_2)$

Overview

Joint representations on $H_1 \otimes H_2$

Observation

For $\varphi_k = \langle \Omega, \pi_k(\cdot)\Omega \rangle$, $\pi_k \colon A_k \to L(H)$, $H_k = \mathbb{C}\Omega \oplus \hat{H}_k$:

$$\varphi_1 \odot \varphi_2 = \langle \Omega, (\pi_1 \odot \pi_2)(\cdot)\Omega \rangle, \quad \text{where}$$

$$\pi_1 \odot \pi_2(a) = \begin{cases} \pi_1(a) \otimes \mathrm{id} & a \in A_1, \odot \in \{\otimes, \triangleleft\} \\ \pi_1(a) \otimes P_\Omega & a \in A_1, \odot \in \{\diamond, \triangleright\} \\ \mathrm{id} \otimes \pi_2(a) & a \in A_2, \odot \in \{\otimes, \triangleright\} \\ P_\Omega \otimes \pi_2(a) & a \in A_2, \odot \in \{\diamond, \triangleleft\} \end{cases}$$

Joint representations on $H_1 \star H_2$

Def: **free product of spaces**

$$H_1 \star H_2 := \bigoplus_{\varepsilon_1 \neq \ldots \neq \varepsilon_n} \hat{H}_{\varepsilon_1} \otimes \cdots \otimes \hat{H}_{\varepsilon_n}$$

Observation

For $\varphi_k = \langle \Omega, \pi_k(\cdot)\Omega \rangle$, $\pi_k \colon A_k \to L(H)$, $H_k = \mathbb{C}\Omega \oplus \hat{H}_k$:

$$\varphi_1 \star \varphi_2 = \langle \Omega, (\pi_1 \star_\ell \pi_2)(\cdot)\Omega \rangle = \langle \Omega, (\pi_1 \star_r \pi_2)(\cdot)\Omega \rangle$$

$$\varphi_1 \diamond \varphi_2 = \langle \Omega, (\pi_1 \diamond \pi_2)(\cdot)\Omega \rangle$$

where

$$\pi_1 \star_\ell \pi_2(a) = \begin{cases} \pi_1(a) \otimes \mathrm{id} & a \in A_1 \\ \pi_2(a) \otimes \mathrm{id} & a \in A_2 \end{cases}, \quad \pi_1 \star_r \pi_2(a) = \begin{cases} \mathrm{id} \otimes \pi_1(a) & a \in A_1 \\ \mathrm{id} \otimes \pi_2(a) & a \in A_2 \end{cases}$$

$$\pi_1 \diamond \pi_2(a) = \pi_k(a) \otimes P_\Omega \quad a \in A_k$$

Benefits of a joint representation

Observation

- tensor, boolean, monotone and antimonotone product can be represented on $H_1 \otimes H_2$
- free and boolean product can be represented on $H_1 \star H_2$
- joint representation helps to prove **positivity**
- joint representation helps to prove **associativity**

Overview

Independence

Fix product operation for *m-faced* algebras $B_i = B_i^{(1)} \sqcup \cdots \sqcup B_i^{(m)}$

$$\bigtimes_i B_i{}' \ni (\varphi_i)_i \mapsto \bigodot_i \varphi_i \in \Big(\bigsqcup_i B_i\Big)'$$

Def: $\odot$-independence of *m-faced* rv's $j_i \colon B_i \to \mathcal{A}$

$$\Phi \circ \bigsqcup_i j_i = \bigodot_i (\Phi \circ j_i)$$

joint distribution = product of marginals

Multi-faced uau-products

Def: *m*-faced universal product (Manzel & Schürmann 2017)

$$B_1' \times B_2' \ni (\varphi_1, \varphi_2) \mapsto \varphi_1 \odot \varphi_2 \in (B_1 \sqcup B_2)'$$

product operation (for arbitrary *m*-faced algebras B_1, B_2) which is

- **u**nital in the sense that $0 \odot \varphi = \varphi = \varphi \odot 0$
- **a**ssociative
- **u**niversal in the sense that (for *m*-faced hom's $j_i : B_i \to A_i$)

$$(\varphi_1 \odot \varphi_2) \circ (j_1 \sqcup j_2) = (\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2)$$

Many faces. . .

Janus, Brahma and Gayatri (images from Wikimedia Commons)

Examples from joint representations

Observation

1. $\exists$ positive 4-faced universal product represented on $H_1 \otimes H_2$
2. $\exists$ positive 3-faced universal product represented on $H_1 \star H_2$

All previous positive examples included

- bimonotone II (G; Gu, Skoufranis & Hasebe): restriction of 1
- bifreeness (Voiculescu): restriction of case 2
- free-boolean, free-free-boolean (Liu): (restriction of) 2
- biboolean, bimonotone I (Gu & Skoufranis + Hasebe): $\ngtr 0$

Overview

Universal products of representations

Def: universal product of $*$-rep's for $\boxtimes \in \{\otimes, \star\}$

$$\text{*-rep}(B_1) \times \text{*-rep}(B_2) \ni (\pi_1, \pi_2) \mapsto \pi_1 \odot \pi_2 \in \text{*-rep}(B_1 \sqcup B_2)$$

product operation which is unital, associative, algebraically universal and **spatially universal**, i.e.

$$\begin{CD} H_k @>{\pi_k(\cdot)}>> H_k \\ @V{W_k}VV @VV{W_k}V \\ G_k @>>{\sigma_k(\cdot)}> G_k \end{CD} \qquad \Longrightarrow \qquad \begin{CD} H_1 \boxtimes H_2 @>{(\pi_1 \odot \pi_2)(\cdot)}>> H_1 \boxtimes H_2 \\ @V{W_1 \boxtimes W_2}VV @VV{W_1 \boxtimes W_2}V \\ G_1 \boxtimes G_2 @>>{(\sigma_1 \odot \sigma_2)(\cdot)}> G_1 \boxtimes G_2 \end{CD}$$

Theorem (G, Hasebe & Ulrich)

m-faced UP of $*$-rep's $\rightsquigarrow$ positive *m*-faced UP of functionals

Universal lifts I

Def: **left universal lift**

family of $*$-hom's $\lambda_{H_1,H_2}\colon L(H_1) \to L(H_1 \boxtimes H_2)$ s.t.

- left restriction property: $\lambda_{H_1,H_2}(T) \upharpoonright H_1 = T$
- left associativity: $\lambda_{H_1,H_2 \boxtimes H_3} = \lambda_{H_1 \boxtimes H_2, H_3} \circ \lambda_{H_1,H_2}$
- left spatial universality:

$$\begin{array}{ccc} H_1 & \xrightarrow{T^{(*)}} & H_1 \\ {\scriptstyle W_1}\big\downarrow & & \big\downarrow{\scriptstyle W_1} \\ G_1 & \xrightarrow[S^{(*)}]{} & G_1 \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} H_1 \boxtimes H_2 & \xrightarrow{\lambda_{H_1,H_2}(T)} & H_1 \boxtimes H_2 \\ {\scriptstyle W_1 \boxtimes W_2}\big\downarrow & & \big\downarrow{\scriptstyle W_1 \boxtimes W_2} \\ G_1 \boxtimes G_2 & \xrightarrow[\lambda_{G_1,G_2}(S)]{} & G_1 \boxtimes G_2. \end{array}$$

Universal lifts II

Def: right universal lift

family of $*$-hom's $\rho_{H_1,H_2}\colon L(H_2) \to L(H_1 \boxtimes H_2)$ s.t.

- right restriction property: $\rho_{H_1,H_2}(T) \upharpoonright H_2 = T$
- right associativity: $\rho_{H_1 \boxtimes H_2, H_3} = \rho_{H_1, H_2 \boxtimes H_3} \circ \rho_{H_2,H_3}$
- right spatial universality:

$$\begin{array}{ccc} H_2 & \xrightarrow{T^{(*)}} & H_2 \\ \downarrow W_2 & & \downarrow W_2 \\ G_2 & \xrightarrow[S^{(*)}]{} & G_2 \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} H_1 \boxtimes H_2 & \xrightarrow{\rho_{H_1,H_2}(T)} & H_1 \boxtimes H_2 \\ \downarrow W_1 \boxtimes W_2 & & \downarrow W_1 \boxtimes W_2 \\ G_1 \boxtimes G_2 & \xrightarrow[\rho_{G_1,G_2}(S)]{} & G_1 \boxtimes G_2. \end{array}$$

Universal lifts III

Def: universal lift

pair (λ, ρ) of left and right universal lift to $\boxtimes$ s.t.

- middle associativity: $\rho_{H_1, H_2 \boxtimes H_3} \circ \lambda_{H_2, H_3} = \lambda_{H_1 \boxtimes H_2, H_3} \circ \rho_{H_1, H_2}$

Theorem (G, Hasebe & Ulrich)

The following objects are in 1-to-1 correspondence:

1. m-faced UP's of $*$-rep's for $\boxtimes$
2. m-tuples of 1-faced UP's of $*$-rep's for $\boxtimes$
3. m-tuples of universal lifts to $\boxtimes$

Overview

A universal deformation

Definition

For $T \in L(H)$, $H = \mathbb{C}\Omega \oplus \hat{H}$ and $\gamma \in \mathbb{T} \cup \{0\}$:

- decompose $T = \begin{pmatrix} \tau & (t')^* \\ t & \hat{T} \end{pmatrix}$ with $\tau \in \mathbb{C}$; $t, t' \in \hat{H}$; $\hat{T} \in L(\hat{H})$
- define $T_\gamma := \begin{pmatrix} |\gamma|\tau & \overline{\gamma}(t')^* \\ \gamma t & |\gamma|\hat{T} \end{pmatrix}$

Lemma

$T \mapsto T_\gamma$ is a $*$-homomorphism

Lifts to the tensor product

Theorem (G, Hasebe & Ulrich)

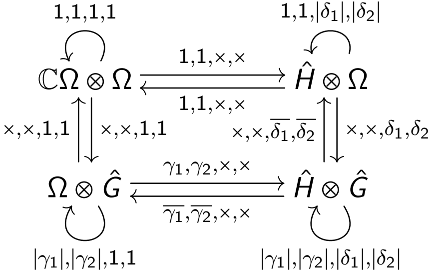
Every universal lift to $\otimes$ is of the form $(\lambda^\gamma, \rho^\delta)$ where

- $\lambda^\gamma(T) = T \otimes P_\Omega + T_\gamma \otimes P_{\Omega^\perp}$
- $\rho^\delta(S) = P_\Omega \otimes S + P_{\Omega^\perp} \otimes S_\delta$
- either $\gamma = \delta \in \mathbb{T}$, or at least one parameter equals zero.

Corollary

$\exists$ uncountably many 2-faced independences represented on $\otimes$

How lifting by $\lambda^{\gamma_1}, \lambda^{\gamma_2}, \rho^{\delta_1}, \rho^{\delta_2}$ creates parameters



2-faced independences represented on $H_1 \otimes H_2$

Parameters that determine the 2-faced universal product ${}_{\gamma_1}^{\gamma_2}\otimes\otimes_{\delta_1}^{\delta_2}$

face 1 \ face 2	tensor ($\gamma_2 = \delta_2 \in \mathbb{T}$)	antimonotone ($\gamma_2 \in \mathbb{T}, \delta_2 = 0$)	monotone ($\gamma_2 = 0, \delta_2 \in \mathbb{T}$)	boolean ($\gamma_2 = \delta_2 = 0$)
tensor ($\gamma_1 = \delta_1 \in \mathbb{T}$)	$\gamma_1\overline{\gamma_2}$	$\gamma_1\overline{\gamma_2}$	$\delta_1\overline{\delta_2}$	$\varnothing$
antimonotone ($\gamma_1 \in \mathbb{T}, \delta_1 = 0$)	$\gamma_1\overline{\gamma_2}$	$\gamma_1\overline{\gamma_2}$	$\gamma_1\overline{\delta_2}$	$\varnothing$
monotone ($\gamma_1 = 0, \delta_1 \in \mathbb{T}$)	$\delta_1\overline{\delta_2}$	$\delta_1\overline{\gamma_2}$	$\delta_1\overline{\delta_2}$	$\varnothing$
boolean ($\gamma_1 = \delta_1 = 0$)	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Lifts to the tensor product – sketch of proof

Lemma 1: Reduction to "boolean creators"

$$\lambda = \lambda' \iff \lambda(a_x^*) = \lambda'(a_x^*) \text{for all } x \in \hat{H}_1$$

$a_x^*(\Omega) = x, a_x^*(x') = 0, \quad a_x^* \in L(\mathbb{C}\Omega, \hat{H}_1) \subset L_\mathrm{a}(H_1)$

(spatial universality + $*$-homomorphism)

Lemma 2: Universal constant

$\exists\ \gamma \in \mathbb{T} \cup \{0\}$ s.t.

$$\lambda(a_x^*) = a_x^* \otimes P_\Omega + \gamma a_x^* \otimes P_{\Omega^\perp}$$

(spatial universality + linearity)

Lifts to the free product

Theorem (G, Hasebe & Ulrich)

Every universal lift to $\star$ is one of the following:

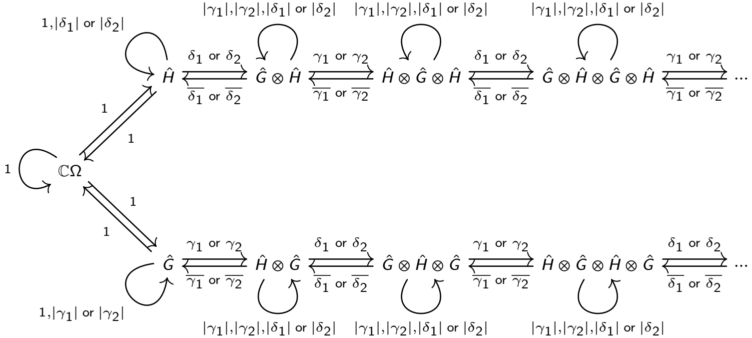
1. $(\overrightarrow{\lambda}^{\gamma}, \overrightarrow{\rho}^{\delta})$ with $\gamma, \delta \in \mathbb{T}$
2. $(\overleftarrow{\lambda}^{\gamma}, \overleftarrow{\rho}^{\delta})$ with $\gamma, \delta \in \mathbb{T}$
3. $(\overrightarrow{\lambda}^{0}, \overrightarrow{\rho}^{0}) = (\overleftarrow{\lambda}^{0}, \overleftarrow{\rho}^{0})$

where

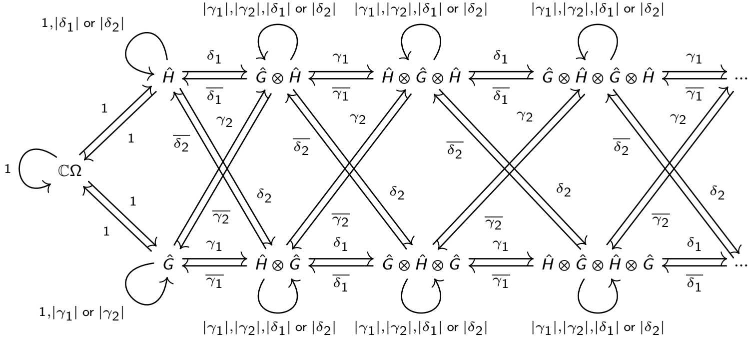
$$\overrightarrow{\lambda}^{\gamma}_{H_1,H_2}(T) = T \otimes P_\Omega + T_\gamma \otimes P_{\Omega^\perp}$$
$$\overrightarrow{\rho}^{\delta}_{H_1,H_2}(S) = S \otimes P_\Omega + S_\delta \otimes P_{\Omega^\perp}$$

and $\overleftarrow{\lambda}^{\gamma}_{H_1,H_2}, \overleftarrow{\rho}^{\delta}_{H_1,H_2}$ analogously from the right.

How lifting by $(\vec{\lambda}^{\gamma_k}, \vec{\rho}^{\delta_k})$ creates parameters



How lifting by $(\overrightarrow{\lambda}^{\gamma_1}, \overrightarrow{\rho}^{\delta_1}), (\overleftarrow{\lambda}^{\gamma_2}, \overleftarrow{\rho}^{\delta_2})$ creates parameters



2-faced independences represented on $H_1 \star H_2$

Parameters that determine the 2-faced UP's ${}_{\gamma_1}^{\gamma_2}\overset{\rightarrow\rightarrow}{*\,*}{}_{\delta_1}^{\delta_2}$ and ${}_{\gamma_1}^{\gamma_2}\overset{\rightarrow\leftarrow}{*\,*}{}_{\delta_1}^{\delta_2}$

face 1 \ face 2	left free $(\vec{\lambda}, \vec{\rho})$ $(\gamma_2, \delta_2 \in \mathbb{T})$	right free $(\overleftarrow{\lambda}, \overleftarrow{\rho})$ $(\gamma_2, \delta_2 \in \mathbb{T})$	boolean $(\gamma_2 = \delta_2 = 0)$
left free $(\vec{\lambda}, \vec{\rho})$ $(\gamma_1, \delta_1 \in \mathbb{T})$	$\gamma_1\overline{\gamma_2}, \delta_1\overline{\delta_2}$	$\gamma_1\overline{\delta_2}, \delta_1\overline{\gamma_2}$	$\varnothing$
boolean $(\gamma_1 = \delta_1 = 0)$	$\varnothing$	$\varnothing$	$\varnothing$

Elements of the proof

A simple linear algebra lemma

Let $\dim H > n$. Then

$$H^{\otimes n} = \operatorname{span}(e_1 \otimes \ldots \otimes e_n : (e_k)_{k=1}^n \text{ orthonormal})$$

The main lemma: existence of universal coefficients

$$\lambda_{H_1,H_2}(a^*_{z_0}) z_1 \otimes z_2 \otimes \cdots \otimes z_n = \sum_{\sigma \in \mathcal{P}^1_{\underline{k}}} c^1_{\underline{k}}(\sigma) z_{\sigma(0)} \otimes z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(n)}.$$

where

- $\mathcal{P}^{k_0}_{\underline{k}} = \{\sigma \in \mathcal{P}(\{0, \ldots, n\}) : k_{\sigma(0)} \neq k_{\sigma(1)} \neq \cdots \neq k_{\sigma(n)}\}$
- $z_i \in \hat{H}_{k_i}$, $k_0 = 1$, $\underline{k} = (k_1 \neq k_2 \ldots \neq k_n)$

Combinatorial approach

Weighted moment-cumulant relation

For fixed $a_1, \dots, a_n \in A$, $a_k \in A^{\mu_k}$, $(\alpha_\pi)_{\pi \in \mathcal{P}(\mu)}$

$$\varphi(a_1 \dots a_n) = \sum_{\pi \in \mathcal{P}(\mu)} \alpha_\pi \prod_{B \in \pi} c_B(a_1, \dots, a_n)$$

defines *cumulants* c_B.

Varšo 2021

List of necessary conditions such that c_B are cumulants of a positive symmetric universal product, most notably:

$$\alpha_\pi = \alpha_{\pi \setminus \{B_1, B_2\} \cup \{B_1 \cup B_2\}} \cdot \alpha_{\{B_1, B_2\}}$$

whenever B_1, B_2 have neighboring legs in the same face.

Open problems

Are there more positive 2-faced independences?

Varšo's combinatorial approach allows

- tensor-free independence
- parameters of modulus < 1

positivity unknown!

Positive UP $\rightsquigarrow$ UP of $*$-rep's?

- other representation spaces besides $H_1 \otimes H_2$, $H_1 \star H_2$?
- more general theory without fixing product of spaces?

Understanding the new independences

- concise mixed-moment and moment-cumulant formulae
- understand limit distributions

Thank you!